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ALGEBRAIC DIFFERENTIAL EQUATIONS FOR ENTIRE HOLOMORPHIC CURVES IN PROJECTIVE HYPERSURFACES OF GENERAL TYPE: OPTIMAL LOWER DEGREE BOUND

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ABSTRACT. Let $X = X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a geometrically smooth projective algebraic complex hypersurface. Using Green-Griffiths jets, we establish the existence of nonzero global algebraic differential equations that must be satisfied by every nonconstant entire holomorphic curve $\mathbb{C} \to X$ if X is of general type, namely if its degree d satisfies the optimal possible lower bound:

$$d \geqslant n+3.$$

The case $n=2$ dates back to Green-Griffiths 1979, while according to very recent advances (Invent. Math. **180**, pp. 161–223, February 2010), the best (and only) lower degree bound known previously in arbitrary dimension n was, using instead Demailly-Semple jets:

$$d \geqslant 2^{n^4}\, n^{4n^3}\, 3^{n^3}\, n^{3n^2}\, (n+1)^{n^2+1}\, n^{2n}\, 12,$$

which, visibly, was far from the conjectured $n+3$.

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§1. INTRODUCTION

Let X be an n-dimensional ($n \geqslant 1$) compact complex manifold and assume it to be *of general type*, i.e., if as usual $K_X = \Lambda^n T_X^*$ denotes its canonical line bundle, assume that the dimension of the space of global pluricanonical sections:

$$h^0\big(X,\, (K_X)^{\otimes m}\big) \geqslant \mathsf{Constant} \cdot m^{\dim X} \qquad (\mathsf{Constant} > 0)$$

grows the fastest it can, as $m \to \infty$, namely the Kodaira dimension of X is maximal equal to n. According to a theorem due to Kodaira, X can then be

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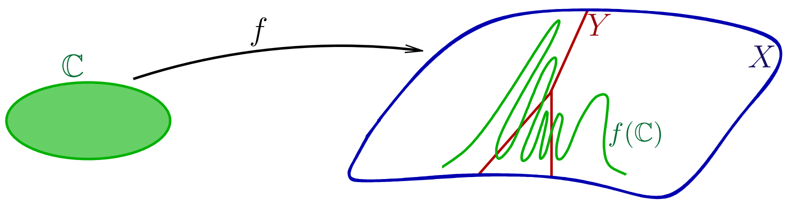
embedded as a geometrically smooth projective algebraic complex manifold in a certain complex projective space $\mathbb{P}^N(\mathbb{C})$. Though it is somewhat delicate to select good embeddings, it is algebraically convenient to view X as being projective *per se*.

In 1979, Green and Griffiths [20] conjectured that there should exist in X a certain *proper* algebraic *sub*variety $Y \subsetneq X$ (possibly with singularities) inside which all nonconstant entire holomorphic curves $f\colon \mathbb{C} \to X$ must necessarily lie, without any such f being allowed to wander anywhere else in $X \backslash Y$.



According to a strategy of thought going back to Bloch, modernized by Green-Griffiths and viewed in a new light by Siu, the 'first half' of this conjecture — so to say — consists in showing that there exist some nonzero global algebraic jet differentials that must be satisfied by every nonconstant entire holomorphic curve $f\colon \mathbb{C} \to X$ (*see* [42, 30, 17] for aspects of the 'second half', not at all considered here).

The principal theorem of this memoir is presented specifically in the case where $X = X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ is a (geometrically smooth) n-dimensional *hypersurface*, because the main mathematical difficulty is essentially to reach arbitrary dimensions $n \geqslant 1$, as was shown recently by the complexity of some of the formal computations sketched in [29, 17] for the case of dimension $n = 4$. But because a substantial part of our proof relies upon works of Brückmann ([5, 6, 7]) which hold in fact for complete intersections, it is very likely that our results may be transferred to such a more general context. Also, one could consider entire holomorphic maps $\mathbb{C}^p \to X^n$ having maximal generic rank p with for any fixed $1 \leqslant p \leqslant n$, as did Pacienza and Rousseau ([32]) recently for $p = 2$ in the case of $X^3 \subset \mathbb{P}^4(\mathbb{C})$. Furthermore, we hope more generally that the techniques developed here could in the future enable us to handle any $X^n \subset \mathbb{P}^N(\mathbb{C})$ of general type having arbitrary codimension $N - n$, but probably requiring more than just general type as a workable assumption ([43]). At least in codimension 1, we are able to gain the following optimal result toward the Green-Griffiths conjecture.

Main Theorem. *Let $X = X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a geometrically smooth n-dimensional projective algebraic complex hypersurface. If X is of general type, namely if its degree d satisfies the optimal lower bound:*

$$d \geqslant n + 3,$$

then there exist global algebraic differential equations on X that must be identically satisfied by every nonconstant entire holomorphic curve $f\colon \mathbb{C} \to X$.

More precisely, if $\mathscr{E}_{\kappa,m}^{GG} T_X^*$ denotes the bundle of Green-Griffiths jet polynomials[1] *of order κ and of weight m over X, then the following holds true.*

Firstly: *for any fixed ample line bundle $\mathscr{A} \to X$ — take* e.g. *simply $\mathscr{A} := \mathscr{O}_X(1)$ —, one has:*

(1)

$$h^0\big(X,\, \mathscr{E}_{\kappa,m}^{GG} T_X^* \otimes \mathscr{A}^{-1}\big) \geqslant \frac{m^{(\kappa+1)n-1}}{(\kappa!)^n\, ((\kappa+1)n-1)!} \Big\{ \tfrac{(\log \kappa)^n}{n!} d(d-n-2)^n - \mathsf{Constant}_{n,d} \cdot (\log \kappa)^{n-1} \Big\} - \\ - \mathsf{Constant}_{n,d,\kappa} \cdot m^{(\kappa+1)n-2},$$

and the right-hand side minorant visibly tends to ∞, as soon as both $\kappa \geqslant \kappa_{n,d}^0$ and $m \geqslant m_{n,d,\kappa}^0$ are large enough.

Secondly: *If P is any global section of $\mathscr{E}_{\kappa,m}^{GG} T_X^* \otimes \mathscr{A}^{-1}$, hence which vanishes on the ample divisor associated to $\mathscr{A}$, then every nonconstant entire holomorphic curve $f \colon \mathbb{C} \to X$ must satisfy the corresponding algebraic differential equation $P(j^\kappa f) = 0$.*

Since the late 1990's, after fundamental works of Bloch, Green-Griffiths and Siu, the so-called Ahlfors-Schwarz for entire holomorphic curves was clarified in full generality, and the second statement above is nowadays (well) known to be a consequence of the first (*see* e.g. Section 7 in [11]).

The case $n = 2$ of this theorem dates back to Green-Griffiths 1979 ([20]). In [36], Rousseau was the first to study effective (Demailly-Semple) jet differentials in dimension 3, under the conditions $d \geqslant 97$. In [14], Diverio treated the next dimensions $n = 4$ and $n = 5$ (improving also $n = 3$ with $d \geqslant 74$), under the conditions $d \geqslant 298$ and $d \geqslant 1222$. In [29], the author of the present article improved for $n = 4$ the lower bound to $d \geqslant 259$. In [15], Diverio showed the (noneffective) existence of a lower bound degree d_n such that $d \geqslant d_n$ insures existence of nonzero global jet differentials. An effective d_n was captured in [17] (*see* $\widetilde{d}_n^1$, p. 192):

$$d \,\geqslant\, 2^{n^4}\, n^{4n^3}\, 3^{n^3}\, n^{3n^2}\, (n+1)^{n^2+1}\, n^{2n}\, 12,$$

far from the optimal $n + 3$, *see* Section 13 for some explanations. Also, one must mention that except in the original Green-Griffiths article [20] for the case $n = 2$, none of the references cited provides explicit effective minorations of h^0 as (1) just above.

In conference talks given in the CIRM (June 2009), in the Hong-Kong University (August 2009) and also later in some seminars (Paris, Marseille, Lyon), the author announced that he was able to gain the Main Theorem under the conjectural assumption that one can majorate the dimensions of the positive cohomology groups of the general Schur bundles[2]:

$$h^q = \dim H^q\big(X,\, \mathscr{S}^{(\ell_1,\dots,\ell_n)} T_X^*\big) \qquad\qquad {\scriptstyle (q=1\cdots n)}$$

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[1] See Sections 2 and 3 for exact definitions.

[2] Definitions and properties may be found in Sections 11 and 4.

over X by a specific formula of the general type:
(2)

$$h^q \leqslant \mathsf{Constant}_n \cdot \big[1 + d + \cdots + d^{n+1}\big] \cdot \prod_{1 \leqslant i < j \leqslant n} \big(\ell_i - \ell_j\big) \cdot \sum_{\substack{\beta_1+\cdots+\beta_{n-1}+\beta_n = n \\ \beta_n \leqslant n-1}} (\ell_1 - \ell_2)^{\beta_1} \cdots (\ell_{n-1} - \ell_n)^{\beta_{n-1}} (\ell_n)^{\beta_n} + \mathsf{Constant}_{n,d} \big(1 + |\ell|^{\frac{n(n+1)}{2} - 1}\big),$$

where $|\ell| = \ell_1 + \cdots + \ell_n$, in which, principally, the exponent β_n of ℓ_n is constrained to be $\leqslant n-1$. In fact, we shall establish that a certain graded bundle $\mathsf{Gr}^\bullet \mathscr{E}_{\kappa,m}^{GG} T_X^*$ introduced by Green and Griffiths themselves which is naturally associated to the jet bundle $\mathscr{E}_{\kappa,m}^{GG} T_X^*$ decomposes as a certain well controlled direct sum of Schur bundles:

$$\mathsf{Gr}^\bullet \mathscr{E}_{\kappa,m}^{GG} T_X^* = \bigoplus_{\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n \geqslant 0} \Big(\mathscr{S}^{(\ell_1,\ell_2,\dots,\ell_n)} T_X^*\Big)^{M_{\ell_1,\ell_2,\dots,\ell_n}^{\kappa,m}},$$

with multiplicities $M_{\ell_1,\ell_2,\dots,\ell_n}^{\kappa,m} \in \mathbb{N}$ understood in combinatorial terms and which are zero for some obvious reasons when $|\ell|$ is less than $\frac{m}{\kappa}$. As is known (cf. Section 2), the cohomology of $\mathscr{E}_{\kappa,m}^{GG} T_X^*$ is controlled (majorated) by the cohomology of $\mathsf{Gr}^\bullet \mathscr{E}_{\kappa,m}^{GG} T_X^*$. The conjectural majoration (2) above was then the most prudent majoration which would insure that each positive cohomology dimension $(1 \leqslant q \leqslant n)$:

$$\begin{aligned} h^q\big(X,\ \mathscr{E}_{\kappa,m}^{GG} T_X^*\big) &\leqslant \sum_{\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n \geqslant 0} M_{\ell_1,\ell_2,\dots,\ell_n}^{\kappa,m} \cdot h^q\big(X,\ \mathscr{S}^{(\ell_1,\ell_2,\dots,\ell_n)} T_X^*\big) \\ &\leqslant \tfrac{m^{(\kappa+1)n-1}}{(\kappa!)^n\, ((\kappa+1)n-1)!} \cdot \mathsf{O}_{n,d}\big(\log(\kappa)^{n-1}\big) + \mathsf{O}_{n,d,\kappa}\big(m^{(\kappa+1)n-2}\big) \end{aligned}$$

(Sections 7, 8, 9 and 10 are devoted to establishing the second inequality) is majorated[3] by a quantity which becomes asymptotically negligible in comparison to the Euler-Poincaré characteristic computed in 1979 by Green and Griffiths:

$$\chi\big(X,\ \mathscr{E}_{\kappa,m}^{GG} T_X^*\big) = \tfrac{m^{(\kappa+1)n-1}}{(\kappa!)^n\, ((\kappa+1)n-1)!} \Big\{ d(d-n-2)^n \, \frac{(\log \kappa)^n}{n!} + \mathsf{O}_{n,d}\big(\log(\kappa)^{n-1}\big) \Big\} + \mathsf{O}_{n,d,\kappa}\big(m^{(\kappa+1)n-2}\big),$$

so that a minoration for h^0 of the sort claimed by the Main Theorem above then immediately follows from $\chi = h^0 - h^1 + \cdots + (-1)^n h^n$ by observing simply:

$$h^0 \geqslant \chi - h^2 - h^4 - h^6 - \cdots .$$

[3] Throughout the paper, we shall sometimes write $\mathsf{O}_{n,d,\kappa}\big(m^{(\kappa+1)n-2}\big)$ to denote a quantity which is majorated by $\mathsf{Constant}_{n,d,\kappa} \cdot m^{(\kappa+1)n-2}$.

But in fact, we shall obtain a majoration better than (2):

$$h^q \leqslant \mathsf{Constant}_{n,d} \prod_{1\leqslant i<j\leqslant n} (\ell_i - \ell_j) \bigg[\sum_{\beta_1'+\cdots+\beta_{n-1}'=n} (\ell_1-\ell_2)^{\beta_1'} \cdots (\ell_{n-1}-\ell_n)^{\beta_{n-1}'} \bigg] + \\ + \mathsf{Constant}_{n,d} \left(1+|\ell|^{\frac{n(n+1)}{2}-1}\right).$$

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§2. Universal combinatorics of Green-Griffiths jets

Algebraic hypersurfaces in a complex projective space. Let $n \geqslant 1$ be a positive integer. On the complex euclidean space $\mathbb{C}^{n+1} \setminus \{0\}$ with the origin deleted, consider so-called *homogeneous coordinates*:

$$[z] := \big[z_0 \colon z_1 \colon \cdots \colon z_n \colon z_{n+1}\big]$$

with the convention that for every nonzero $c \in \mathbb{C} \setminus \{0\}$, any $[cz]$ all of which coordinates are equally c-dilated is equivalent to $[z]$:

$$\big[cz_0 \colon cz_1 \colon \cdots \colon cz_n \colon cz_{n+1}\big] = \big[z_0 \colon z_1 \colon \cdots \colon z_n \colon z_{n+1}\big].$$

The set of such $\big[z_0 \colon \cdots \colon z_{n+1}\big]$ constitutes the so-called *complex projective space* of dimension $n+1$:

$$\begin{aligned}\mathbb{P}_{n+1}(\mathbb{C}) &:= \big\{\big[z_0 \colon z_1 \colon \cdots \colon z_n \colon z_{n+1}\big]\big\} \\ &= \big\{z = (z_0, z_1, \ldots, z_n, z_{n+1}) \in \mathbb{C}^{n+1} \setminus \{0\}\big\} \big/ \big(z \sim \mathbb{C}^* z\big),\end{aligned}$$

and is a compact complex manifold. Consider now a complex projective algebraic hypersurface:

$$X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$$

of dimension n given as the zero-set:

$$X = \Big\{ \big[z_0 \colon z_1 \colon \ldots \colon z_n \colon z_{n+1}\big] \in \mathbb{P}^{n+1}(\mathbb{C}) \colon \\ P(z_0, z_1, \ldots, z_n, z_{n+1}) = 0 \Big\}.$$

of some polynomial which is homogeneous of a certain degree $d \geqslant 1$, say of the general form:

$$P := \sum_{\beta_0+\beta_1+\cdots+\beta_n+\beta_{n+1}=d} \mathsf{coeff} \cdot z_0^{\beta_0} z_1^{\beta_1} \cdots z_n^{\beta_n} z_{n+1}^{\beta_{n+1}}$$

with fixed complex coefficients $P_{\beta_0,\beta_1,\dots,\beta_n,\beta_{n+1}}$, one of them at least being nonzero. We assume throughout that X is geometrically smooth, namely the differential of P is nonzero at every point of X.

Jets of holomorphic discs in X. In a local chart $(X, x_0) \simeq (\mathbb{C}^n, 0)$ centered at a point $x_0 \in X$ equipped with n complex coordinates $(x_1, \dots, x_n)$, one looks at holomorphic discs passing through x_0:

$$f : (\mathbb{D}, 0) \to (\mathbb{C}^n, 0) \simeq (X, x_0),$$

namely with $f(0) = x_0$, which possess of course n components:

$$\big(f_1(\zeta), f_2(\zeta), \dots, f_n(\zeta)\big).$$

For any integer $\kappa \geqslant 1$, the associated κ-jet map of any such a holomorphic disc gathers all its $n\kappa$ derivatives up to order κ with respect to the (single) source variable $\zeta \in \mathbb{D}$:

$$j^\kappa f(\zeta) = \big(f_1', \dots, f_n', f_1'', \dots, f_n'', \dots\dots, f_1^{(\kappa)}, \dots, f_n^{(\kappa)}\big)(\zeta).$$

Accordingly, one is led to introduce $n\kappa$ new independent *jet coordinates* that will simply be denoted as:

$$\big(x_1', \dots, x_n',\, x_1'', \dots, x_n'',\, \dots\dots,\, x_n^{(\kappa)}, \dots, x_n^{(\kappa)}\big),$$

so that $\big(x, x', x'', \dots, x^{(\kappa)}\big)$ provide $n+n\kappa$ coordinates on the space of *un*centered κ-jets of maps $\mathbb{D} \to X$.

Weighted homogeneous jet polynomials. Above any fixed point $x_0 \in X$, Green-Griffiths ([20]) introduced a certain "fiber" which consists of all polynomials in these jet variables $x_i^{(\lambda)}$, $1 \leqslant i \leqslant n$, $1 \leqslant \lambda \leqslant \kappa$, that are of the following type:

(3)

$$\mathsf{P}\big(x', x'', \dots, x^{(\kappa)}\big) = \sum_{\substack{\alpha_1,\alpha_2,\dots,\alpha_\kappa \in \mathbb{N}^n \\ |\alpha_1|+2|\alpha_2|+\cdots+\kappa|\alpha_\kappa|=m}} \mathsf{coeff}_{\alpha_1,\alpha_2,\dots,\alpha_\kappa} \cdot \big(x'\big)^{\alpha_1} \big(x''\big)^{\alpha_2} \cdots \big(x^{(\kappa)}\big)^{\alpha_\kappa},$$

where $m \geqslant 1$ is some integer, where $\alpha_\lambda = (\alpha_{\lambda,1}, \dots, \alpha_{\lambda,n}) \in \mathbb{N}^n$ for $1 \leqslant \lambda \leqslant \kappa$ are multiindices of length $|\alpha_\lambda| = \alpha_{\lambda,1} + \cdots + \alpha_{\lambda,n}$, and where $\mathsf{coeff}_{\alpha_1,\alpha_2,\dots,\alpha_\kappa}$ are arbitrary complex coefficients, or equivalently if written in greater length:

$$\sum_{\substack{\alpha_1,\alpha_2,\dots,\alpha_\kappa \in \mathbb{N}^n \\ |\alpha_1|+2|\alpha_2|+\cdots+\kappa|\alpha_\kappa|=m}} \mathsf{coeff}_{\alpha_1,\alpha_2,\dots,\alpha_\kappa} \cdot \prod_{1\leqslant i\leqslant n} \big(x_i'\big)^{\alpha_{1,i}} \prod_{1\leqslant i\leqslant n} \big(x_i''\big)^{\alpha_{2,i}} \cdots \prod_{1\leqslant i\leqslant n} \big(x_i^{(\kappa)}\big)^{\alpha_{\kappa,i}}.$$

Visibly, such polynomials enjoy weighted homogeneity:

$$\mathsf{P}\big(\delta x',\, \delta^2 x'',\, \dots,\, \delta^\kappa x^{(\kappa)}\big) = \delta^m\, \mathsf{P}\big(x',\, x'',\, \dots,\, x^{(\kappa)}\big)$$

of the fixed weight m with respect to the anisotropic complex jet dilation defined by:

$$\delta \cdot \big(x'_{i_1}, x''_{i_2}, \dots, x^{(\kappa)}_{i_\kappa}\big) := \big(\delta\, x'_{i_1}, \delta^2 x''_{i_2}, \dots, \delta^\kappa x^{(\kappa)}_{i_\kappa}\big), \qquad\qquad \delta \in \mathbb{C},$$

whence for memory in all what follows one sees that:

$$m = \mathsf{weight} = \text{(fixed) total number of appearing ``primes''}.$$

Lemma. *As the point x_0 runs in X, these polynomial fibers organize coherently as a* holomorphic vector *bundle $\mathscr{E}^{GG}_{\kappa,m} T^*_X$ over X of rank equal to the number of arbitrary coefficients* $\mathsf{coeff}_{\alpha_1,\dots,\alpha_\kappa}$*, namely to:*

$$\mathrm{Card}\,\big\{(\alpha_1, \alpha_2, \dots, \alpha_\kappa) \in (\mathbb{N}^n)^\kappa \colon |\alpha_1| + 2|\alpha_2| + \cdots + \kappa|\alpha_\kappa| = m\big\}.$$

Proof. Consider an arbitrary change of (local) holomorphic chart on X:

$$(x_1, \dots, x_n) \longmapsto (y_1, \dots, y_n) = \big(\Psi_1(x_1, \dots, x_n),\ \dots\dots,\ \Psi_n(x_1, \dots, x_n)\big),$$

understood as inducing a change of local trivialization for the bundle. One must establish that the new coefficients of the transformed jet polynomials express *linearly* in terms of the coefficients of (3).

To begin with, the knowledge of how the $y_j^{(\lambda)}$ express in terms of the $x_i^{(\tau)}$ is provided by an application of the chain rule for the differentiation of a composed holomorphic disc $\zeta \mapsto \Psi\big(f_1(\zeta), \dots, f_n(\zeta)\big)$. The closed combinatorial formula writes as follows, for any λ with $1 \leqslant \lambda \leqslant \kappa$ and for any j with $1 \leqslant j \leqslant n$.

Theorem. (see [8, 27]) *The λ-jet of $\Psi_j(f_1, \dots, f_n)$ is given by the following multivariate Faà di Bruno formula, written without the argument ζ:*

$$\begin{aligned}\big[\Psi_j(f_1, \dots, f_n)\big]^{(\lambda)} = \sum_{e=1}^{\lambda} \sum_{1 \leqslant \tau_1 < \cdots < \tau_e \leqslant \lambda} \sum_{\mu_1 \geqslant 1, \dots, \mu_e \geqslant 1} \sum_{\mu_1 \tau_1 + \cdots + \mu_e \tau_e = \lambda} & \frac{\lambda!}{(\tau_1!)^{\mu_1} \mu_1! \cdots (\tau_e!)^{\mu_e} \mu_e!} \\ \sum_{j_1^1, \dots, j_{\mu_1}^1 = 1}^{n} \cdots\cdots \sum_{j_1^e, \dots, j_{\mu_e}^e = 1}^{n} & \frac{\partial^{\mu_1 + \cdots + \mu_e} \Psi_j}{\partial x_{j_1^1} \cdots \partial x_{j_{\mu_1}^1} \cdots\cdots \partial x_{j_1^e} \cdots \partial x_{j_{\mu_e}^e}} \cdot \\ & \cdot f_{j_1^1}^{(\tau_1)} \cdots f_{j_{\mu_1}^1}^{(\tau_1)} \cdots\cdots f_{j_1^e}^{(\tau_e)} \cdots f_{j_{\mu_e}^e}^{(\tau_e)}.\end{aligned}$$

To read this general formula, we comment it backward, understanding it rather as a (polynomial, invertible) transformation between independent jet variables:

$$\begin{aligned}y_j^{(\lambda)} = \sum_{e=1}^{\lambda} \sum_{1 \leqslant \tau_1 < \cdots < \tau_e \leqslant \lambda} \sum_{\mu_1 \geqslant 1, \dots, \mu_e \geqslant 1} \sum_{\mu_1 \tau_1 + \cdots + \mu_e \tau_e = \lambda} & \frac{\lambda!}{(\tau_1!)^{\mu_1} \mu_1! \cdots (\tau_e!)^{\mu_e} \mu_e!} \\ \sum_{j_1^1, \dots, j_{\mu_1}^1 = 1}^{n} \cdots\cdots \sum_{j_1^e, \dots, j_{\mu_e}^e = 1}^{n} & \frac{\partial^{\mu_1 + \cdots + \mu_e} \Psi_j}{\partial x_{j_1^1} \cdots \partial x_{j_{\mu_1}^1} \cdots\cdots \partial x_{j_1^e} \cdots \partial x_{j_{\mu_e}^e}} \cdot \\ & \cdot x_{j_1^1}^{(\tau_1)} \cdots x_{j_{\mu_1}^1}^{(\tau_1)} \cdots\cdots x_{j_1^e}^{(\tau_e)} \cdots x_{j_{\mu_e}^e}^{(\tau_e)}.\end{aligned}$$

The general monomial $\prod x_{\bullet}^{(\tau_1)} \prod x_{\bullet}^{(\tau_2)} \cdots \prod x_{\bullet}^{(\tau_2)}$ in the jet variables gathers derivatives of increasing orders $\tau_1 < \tau_2 < \cdots < \tau_e$ with $\mu_1, \mu_2, \dots, \mu_e$ counting their respective numbers. Then Ψ_j is subjected to a partial derivative of order $\mu_1 + \mu_2 + \cdots + \mu_e$, the total number of letters $x_{\bullet}^{\bullet}$ in the monomial in question. Because there are $n+1$ variables x_i, the dots in the $x_{\bullet}^{(\tau_c)}$ should receive indices, and in fact, there appear general sums $\sum_{j_1^c, \dots, j_{\mu_c}^c = 1}^{n}$ over *all possible* such indices.

This precise closed combinatorial formula is not really needed for the proof of our lemma, and instead, it is sufficient to know that each $y_j^{(\lambda)}$ is a certain polynomial in the $x_i^{(\tau_c)}$, with coefficients depending linearly upon the λ-jet of Ψ, the weight $\mu_1\tau_1 + \cdots + \mu_e\tau_e$ of each appearing monomial $x_{j_1^1}^{(\tau_1)} \cdots x_{j_{\mu_1}^1}^{(\tau_1)} \cdots \cdots x_{j_1^e}^{(\tau_e)} \cdots x_{j_{\mu_e}^e}^{(\tau_e)}$ being constant equal to λ, and this fact is easily proved by a rough induction argument. We can abbreviate this as:

$$y_j^{(\nu)} = \sum_{i_1=1}^{n} \frac{\partial \Psi_j}{\partial x_{i_1}} + \cdots + \sum_{i_1, \dots, i_\lambda = 1}^{n} \frac{\partial^\lambda \Psi_j}{\partial x_{i_1} \cdots \partial x_{i_\lambda}}$$

Consequently, we must, as said, examine how an y-jet general polynomial of weight m like the x-jet polynomial P in (3):

$$\mathsf{Q}\big(y', y'', \dots, y^{(\kappa)}\big) = \sum_{\substack{\alpha_1, \alpha_2, \dots, \alpha_\kappa \in \mathbb{N}^n \\ |\alpha_1| + 2|\alpha_2| + \cdots + \kappa|\alpha_\kappa| = m}} \mathsf{Q}_{\alpha_1, \alpha_2, \dots, \alpha_\kappa} \cdot \big(y'\big)^{\alpha_1} \big(y''\big)^{\alpha_2} \cdots \big(y^{(\kappa)}\big)^{\alpha_\kappa}$$

is transformed. From the theorem stated above (or from the rough induction argument), we deduce that a general monomial of a weight m:

$$\begin{aligned}
&\big(y'\big)^{\alpha_1} \big(y''\big)^{\alpha_2} \cdots \big(y^{(\lambda)}\big)^{\alpha_\lambda} \cdots \big(y^{(\kappa)}\big)^{\alpha_\kappa} = \\
&= \Big(\sum_{i_1} \Psi_{x_{i_1}} \, x'_{i_1} \Big)^{\alpha_1} \Big(\sum_{i_1} \Psi_{x_{i_1}} \, x''_{i_1} + \sum_{i_1, i_2} \Psi_{x_{i_1} x_{i_2}} \, x'_{i_1} x'_{i_2} \Big)^{\alpha_2} \cdots \\
&\cdots \Big(\sum_{i_1} \Psi_{x_{i_1}} \, x_{i_1}^{(\lambda)} + \cdots + \sum_{i_1, \dots, i_\lambda} \Psi_{x_{i_1} \dots x_{i_\lambda}} \, x'_{i_1} \dots x'_{i_\lambda} \Big)^{\alpha_\lambda} \cdots \\
&\cdots \Big(\sum_{i_1} \Psi_{x_{i_1}} \, x_{i_1}^{(\kappa)} + \cdots + \sum_{i_1, \dots, i_\kappa} \Psi_{x_{i_1} \dots x_{i_\kappa}} \, x'_{i_1} \dots x'_{i_\kappa} \Big)^{\alpha_\kappa}
\end{aligned} \tag{4}$$

is clearly transformed to a jet polynomial of weight m:

$$\big(y'\big)^{\alpha_1} \cdots \big(y^{(\kappa)}\big)^{\alpha_\kappa} = \sum_{|\beta_1| + \cdots + \kappa|\beta_\kappa| = m} \mathsf{H}_{\beta_1, \dots, \beta_\kappa}^{\alpha_1, \dots, \alpha_\kappa} \big(j^\kappa \Psi\big) \cdot (x')^{\beta_1} \cdots (x^{(\kappa)})^{\beta_\kappa}$$

having coefficients that are certain universal polynomials in the κ-jet of Ψ. It therefore follows that $\mathsf{Q}(y', \dots, y^{(\kappa)})$ is transformed to:

$$\begin{aligned}
&\sum_{|\alpha_1| + \cdots + \kappa|\alpha_\kappa| = m} \; \sum_{|\beta_1| + \cdots + \kappa|\beta_\kappa| = m} \mathsf{Q}_{\alpha_1, \dots, \alpha_\kappa} \cdot \mathsf{H}_{\beta_1, \dots, \beta_\kappa}^{\alpha_1, \dots, \alpha_\kappa} \big(j^\kappa \Psi\big) \cdot (x')^{\beta_1} \cdots (x^{(\kappa)})^{\beta_\kappa} \\
&=: \sum_{|\beta_1| + \cdots + \kappa|\beta_\kappa| = m} \mathsf{P}_{\beta_1, \dots, \beta_\kappa} \cdot (x')^{\beta_1} \cdots (x^{(\kappa)})^{\beta_\kappa}
\end{aligned}$$

with the following *linear* relationship between coefficients:

$$\mathsf{P}_{\beta_1,\dots,\beta_\kappa} = \sum_{|\alpha_1|+\cdots+\kappa|\alpha_\kappa|=m} \mathsf{H}^{\alpha_1,\dots,\alpha_\kappa}_{\beta_1,\dots,\beta_\kappa}\big(j^\kappa\Psi\big)\cdot\mathsf{Q}_{\alpha_1,\dots,\alpha_\kappa}.$$

This shows that $\mathscr{E}^{GG}_{\kappa,m}T^*_X$ effectively is a vector bundle, because the cocycle relations and the inverse trivializations follow from the transitivity and from the invertibility of change of local coordinates on X. □

Symmetric pluri-tensor decomposition. As is known in the domain ([20, 11, 17]), a certain graded holomorphic vector bundle $\mathsf{Gr}^\bullet\mathscr{E}^{GG}_{\kappa,m}T^*_X$ naturally associated to this Green-Griffiths bundle $\mathscr{E}^{GG}_{\kappa,m}T^*_X$ happens to decompose into the following direct sum of multi-tensored symmetric powers of the cotangent bundle T^*_X of X:

$$\mathsf{Gr}^\bullet\mathscr{E}^{GG}_{\kappa,m}T^*_X = \bigoplus_{\ell_1+2\ell_2+\cdots+\kappa\ell_\kappa=m} \mathrm{Sym}^{\ell_1}T^*_X\otimes\mathrm{Sym}^{\ell_2}T^*_X\otimes\cdots\otimes\mathrm{Sym}^{\ell_\kappa}T^*_X.$$

Informally speaking, such a decomposition just relates to the fact that the general m-weighted polynomial (3) on p. 6 looks like a linear combination of (tensor!) products of the (individually symmetric!) monomials $(x')^{\alpha_1}$, $(x'')^{\alpha_2}$, ..., $(x^{(\kappa)})^{\alpha_\kappa}$ with, say for a good correspondence:

$$\ell_1\equiv|\alpha_1|,\qquad \ell_2\equiv|\alpha_2|,\qquad \dots\dots,\qquad \ell_\kappa\equiv|\alpha_\kappa|.$$

But this view is not rigorous, so let us explain with more details than in [20, 11, 17] how one builds the graded bundle $\mathsf{Gr}^\bullet\mathscr{E}^{GG}_{\kappa,m}T^*_X$.

Consider again the transformation (4). If $\alpha_{\lambda+1}=\cdots=\alpha_\kappa=0$ for some λ, one sees that after expansion, the total number $|\alpha_1|+2|\alpha_2|+\cdots+\lambda|\alpha_\lambda|$ of primes remains unchanged. However, if there exists some μ with $\lambda+1\leqslant\mu\leqslant\kappa$ such that $\alpha_\mu\neq 0$, then in general the expansion of the factor $\big(\Psi(x)^{(\mu)}\big)^{\alpha_\mu}$ adds a total of $\mu|\alpha_\mu|$ further primes to the monomials in x', x'', ..., $x^{(\lambda)}$ that was already obtained by expanding the first λ factors $\big(\Psi(x)'\big)^{\alpha_1}\cdots\big(\Psi(x)^{(\lambda)}\big)^{\alpha_\lambda}$. Thus in all cases, after an arbitrary change of coordinates $x\mapsto\Psi(x)$, the λ-restricted weight $|\alpha_1|+2|\alpha_2|+\cdots+\lambda|\alpha_\lambda|$ *can only increase*. Following [20], one may hence define for any λ fixed in advance with $1\leqslant\lambda\leqslant\kappa$ a (decreasing) *filtered sequence*:

$$\mathscr{E}^{GG}_{\kappa,m}T^*_X = \mathscr{F}^0_\lambda\supset\mathscr{F}^1_\lambda\supset\mathscr{F}^2_\lambda\supset\cdots\supset\mathscr{F}^m_\lambda\supset\{0\}=\mathscr{F}^{m+1}_\lambda$$

of subbundles of $\mathscr{E}^{GG}_{\kappa,m}T^*_X$ whose pieces for any $q=1,2,\dots,m$ are naturally defined by:

$$\mathscr{F}^q_\lambda=\mathscr{F}^q_\lambda\big(\mathscr{E}^{GG}_{\kappa,m}T^*_X\big)=\left\{\begin{array}{l}\mathsf{P}\big(x',\dots,x^{(\lambda)},\dots,x^{(\kappa)}\big)\in\mathscr{E}^{GG}_{\kappa,m}T^*_X\text{ involving only monomials}\\ (x')^{\alpha_1}\cdots(x^{(\lambda)})^{\alpha_\lambda}\cdots(x^{(\kappa)})^{\alpha_\kappa}\text{ with }|\alpha_1|+\cdots+\lambda|\alpha_\lambda|\geqslant q\end{array}\right\}.$$

Notice that $\mathscr{F}^m_\lambda=\mathscr{E}^{GG}_{\lambda,m}T^*_X$. If we now set $\lambda=\kappa-1$, the graded bundle associated with this filtration:

$$\mathsf{Gr}^\bullet\mathscr{E}^{GG}_{\kappa,m}T^*_X=\big(\mathscr{F}^0_{\kappa-1}/\mathscr{F}^1_{\kappa-1}\big)\oplus\cdots\oplus\big(\mathscr{F}^{m-1}_{\kappa-1}/\mathscr{F}^m_{\kappa-1}\big)\oplus\mathscr{E}^{GG}_{\kappa-1,m}T^*_X$$

is constituted of quotient factors:

$$\mathscr{G}_{\kappa-1}^{q} := \mathscr{F}_{\kappa-1}^{q} \big/ \mathscr{F}_{\kappa-1}^{q+1} \qquad\qquad {\scriptstyle (q\,=\,0\,\cdots\,m\,-\,1)}$$

which consist of polynomials P as above for which:

$$|\alpha_1| + \cdots + (\kappa-1)|\alpha_{\kappa-1}| = q$$

modulo polynomials for which $|\alpha_1| + \cdots + (\kappa-1)|\alpha_{\kappa-1}| \geqslant q+1$. It follows at once that $q + \kappa|\alpha_\kappa| = m$ in such polynomials, that is to say:

$$q = m - \kappa\ell_\kappa$$

for some integer $\ell_\kappa \in \mathbb{N}$ with $\mathsf{Ent}\big[\frac{m}{\kappa}\big] \geqslant \ell_\kappa \geqslant 0$. In particular, this quotient $\mathscr{F}_{\kappa-1}^{q} \big/ \mathscr{F}_{\kappa-1}^{q+1}$ reduces to $\{0\}$ whenever $m-q$ is *not* divisible by κ.

We now claim that:

$$\mathscr{G}_{\kappa-1}^{m-\kappa\ell_\kappa}\big(\mathscr{E}_{\kappa,m}^{GG}T_X^*\big) \simeq \mathscr{E}_{\kappa-1,\,m-\kappa\ell_\kappa}^{GG}T_X^* \otimes \operatorname{Sym}^{\ell_\kappa}T_X^*.$$

Indeed, under an arbitrary change of coordinates $x \mapsto \Psi(x) = y$, the constituents $x^{(\kappa)}$ of the monomials of highest jet $\big(x^{(\kappa)}\big)^{\alpha_\kappa}$ with $|\alpha_\kappa| = \ell_\kappa$ are transformed to:

$$y^{(\kappa)} = \sum_{i_1=1}^{n} \Psi_{x_{i_1}}\, x_{i_1}^{(\kappa)} \ \text{ modulo } \big(x', \dots, x^{(\kappa-1)}\big),$$

so that they visibly transform in exactly the same covariant way as the covectors in T_X^*, namely:

$$d\Psi = \sum_{i_1=1}^{n} \Psi_{x_{i_1}}\, dx_{i_1},$$

and so, the $\big(x^{(\kappa)}\big)^{\alpha_\kappa}$ transform as $\operatorname{Sym}^{\ell_\kappa}T_X^*$. The other constituents of $\mathscr{F}_{\kappa-1}^{m-\kappa\ell_\kappa}$ depend only on the $(\kappa-1)$-jet and are of the remaining weight $m-\kappa\ell_\kappa$, whence the claimed isomorphism follows.

Putting together all these isomorphisms, we get:

$$\begin{aligned}
\mathsf{Gr}^\bullet \mathscr{E}_{\kappa,m}^{GG}T_X^* &= \bigoplus_{\mathsf{Ent}[\frac{m}{\kappa}]\geqslant\ell_\kappa\geqslant 1} \mathscr{G}_{\kappa-1}^{m-\kappa\ell_\kappa} \bigoplus \mathscr{E}_{\kappa-1,m}^{GG}T_X^* \\
&= \bigoplus_{\mathsf{Ent}[\frac{m}{\kappa}]\geqslant\ell_\kappa\geqslant 1} \Big(\mathscr{E}_{\kappa-1,\,m-\kappa\ell_\kappa}^{GG}T_X^* \otimes \operatorname{Sym}^{\ell_\kappa}T_X^*\Big) \bigoplus \big(\mathscr{E}_{\kappa-1,m}^{GG}T_X^* \otimes \operatorname{Sym}^0 T_X^*\big) \\
&= \bigoplus_{\mathsf{Ent}[\frac{m}{\kappa}]\geqslant\ell_\kappa\geqslant 0} \mathscr{E}_{\kappa-1,\,m-\kappa\ell_\kappa}^{GG}T_X^* \otimes \operatorname{Sym}^{\ell_\kappa}T_X^*.
\end{aligned}$$

Now an induction of this isomorphism applied to $\mathsf{Gr}^\bullet\big(\mathscr{E}_{\kappa-1,\,m-\kappa\ell_\kappa}^{GG}T_X^*\big)$ yields the announced decomposition for $\mathsf{Gr}^\bullet\mathscr{E}_{\kappa,m}^{GG}T_X^*$.

Theorem. *The holomorphic vector bundle $\mathscr{E}_{\kappa,m}^{GG}T_X^*$ of Green-Griffiths polynomials of weight m in the κ-jet of local complex curves $\mathbb{D} \to X$ admits a natural*

filtration whose associated graded bundle is isomorphic to the following direct sum of multi-tensored symmetric powers of the cotangent bundle:

$$\boxed{\mathsf{Gr}^\bullet \mathscr{E}_{\kappa,m}^{GG} T_X^* = \bigoplus_{\ell_1+2\ell_2+\cdots+\kappa\ell_\kappa=m} \mathrm{Sym}^{\ell_1} T_X^* \otimes \mathrm{Sym}^{\ell_2} T_X^* \otimes \cdots \otimes \mathrm{Sym}^{\ell_\kappa} T_X^*}.$$

Furthermore, for every $q = 1, 2, \ldots, n$, one has the inequalities:

$$\dim H^q\big(X,\, \mathscr{E}_{\kappa,m}^{GG} T_X^*\big) \leqslant \sum_{\ell_1+2\ell_2+\cdots+\kappa\ell_\kappa=m} \dim H^q\Big(X,\ \mathrm{Sym}^{\ell_1} T_X^* \otimes \mathrm{Sym}^{\ell_2} T_X^* \otimes \cdots \otimes \mathrm{Sym}^{\ell_\kappa} T_X^*\Big).$$

To complete the proof, it only remains to check the inequality between the cohomology dimensions. Let us consider instead in greater generality the following situation, which clearly embraces the last claim above.

Lemma. *Suppose a holomorphic vector bundle $E \to X$ admits a filtration:*

$$\{0\} = E_{r+1} \subset E_r \subset E_{r-1} \subset \cdots \subset E_{k+1} \subset E_k \subset E_{k-1} \subset \cdots \subset E_1 \subset E_0 = E$$

by nested holomorphic subbundles E_k, the associated graded bundle being:

$$\mathrm{Gr}^\bullet E = E_r \oplus \big(E_{r-1}/E_r\big) \oplus \cdots \oplus \big(E_k/E_{k+1}\big) \oplus \big(E_{k-1}/E_k\big) \oplus \cdots \oplus \big(E_0/E_1\big).$$

where, for a good notational correspondence: $\mathsf{Gr}^k E := E_k/E_{k+1}$. Then for every $q = 0, 1, \ldots, n$, the following inequality between cohomological dimensions holds:

$$\dim H^q\big(X,\ E\big) \leqslant \sum_{k=0}^{r} \dim H^q\big(X,\ E_k/E_{k+1}\big).$$

Proof. To each obviously true short exact sequence:

$$(5) \qquad 0 \longrightarrow \mathsf{Gr}^k E \longrightarrow E/E_{k+1} \longrightarrow E/E_k \longrightarrow 0 \qquad\qquad {\scriptstyle (k\,=\,0,\,1,\,\ldots,\,r)}$$

is associated the long exact sequence between cohomology groups:

$$(6) \qquad \cdots \longrightarrow H^q\big(X,\ \mathsf{Gr}^k E\big) \longrightarrow H^q\big(X,\ E/E_{k+1}\big) \longrightarrow H^q\big(X,\ E/E_k\big) \longrightarrow \cdots,$$

and the trivial majoration: $\dim B \leqslant \dim A + \dim C$ of the dimension of any member B of any long exact sequence of vector spaces by the sum of the dimensions of its two immediate neighbors gives us here:

$$\dim H^q\big(X,\ E/E_{k+1}\big) \leqslant \dim H^q\big(X,\ \mathsf{Gr}^k E\big) + \dim H^q\big(X,\ E/E_k\big) \qquad\qquad {\scriptstyle (k\,=\,0,\,1,\,\ldots,\,r)}.$$

Starting from $k = 0$ for which $\mathsf{Gr}^0 E = E/E_1$ and $E/E_0 = \{0\}$, a plain summation up to $k = r$ of these inequalities cancels out all factors involving E/E_k except only one on the left: $E/E_{r+1} = E$, and we get the desired inequality:

$$\dim H^q\big(X,\ E\big) = \dim H^q\big(X,\ E/E_{r+1}\big) \leqslant \sum_{k=0}^{r} \dim H^q\big(X,\ \mathsf{Gr}^k E\big)$$

which, when applied to the Green-Griffiths bundle, terminates our detailed restitution of the theorem. □

Furthermore, in specific situations where the dimensions of the first cohomology groups of the graded pieces E_k/E_{k+1} do not vanish but happen to become asymptotically (much) smaller than the dimensions of their zeroth cohomology groups, a useful second lemma is as follows.

Lemma. *Under the same assumptions and just for* $i = 0$*, in addition to the above majoration* $h^0(X,E) \leqslant \sum_{k=0}^r h^0(X,\, E_k/E_{k+1})$*, one has the* minoration:

$$h^0(X,\, E) \geqslant \sum_{k=0}^{r} h^0\big(X,\, E_k/E_{k+1}\big) - \sum_{k=0}^{r} h^1\big(X,\, E_k/E_{k+1}\big).$$

Proof. As a preliminary, we observe that any long exact sequence

$$0 \longrightarrow A \overset{a}{\longrightarrow} B \overset{b}{\longrightarrow} C \overset{c}{\longrightarrow} D \overset{d}{\longrightarrow} \cdots$$

can be stopped at its fourth term by replacing it with the four-terms sequence:

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow \mathrm{Im}(c) \longrightarrow 0$$

which is easily checked to be also exact. Then by considering the basic equality which comes out by alternately summing the dimensions of its members:

$$0 = \dim(A) - \dim(B) + \dim(C) - \dim(\mathrm{Im}(c)),$$

we deduce from the trivial inequality $\dim(\mathrm{Im}(c)) \leqslant \dim(D)$, the useful minoration:

$$\dim(B) \geqslant \dim(A) + \dim(C) - \dim(D).$$

Applying now such an inequality to the first four terms of the long exact sequence associated to the k-th quotient exact sequence (5) above:

$$0 \longrightarrow H^0\big(X,\, \mathsf{Gr}^k E\big) \longrightarrow H^0\big(X,\, E/E_{k+1}\big) \longrightarrow H^0\big(X,\, E/E_k\big) \longrightarrow H^1\big(X,\, \mathsf{Gr}^k E\big) \longrightarrow \cdots,$$

we readily deduce:

$$h^0\big(X,\, E/E_{k+1}\big) \geqslant h^0\big(X,\, \mathsf{Gr}^k E\big) + h^0\big(X,\, E/E_k\big) - h^1\big(X,\, \mathsf{Gr}^k E\big).$$

Starting then from $k = 0$ for which $\mathsf{Gr}^0 E = E/E_1$ and $E/E_0 = \{0\}$, a plain summation of these inequalities up to $k = r$ cancels out all terms involving an E/E_k except one: $E/E_{r+1} = E$, and we get the announced minoration. $\square$

As a corollary, by applying this second elementary lemma to the Green-Griffiths bundle, we gain a possibly useful general minoration:

(7)
$$\begin{aligned} h^0\big(X,\, \mathscr{E}^{GG}_{\kappa,m} T_X^*\big) \geqslant & \sum_{\ell_1+2\ell_2+\cdots+\kappa\ell_\kappa=m} h^0\Big(X,\, \mathrm{Sym}^{\ell_1} T_X^* \otimes \mathrm{Sym}^{\ell_2} T_X^* \otimes \cdots \otimes \mathrm{Sym}^{\ell_\kappa} T_X^*\Big) - \\ & - \sum_{\ell_1+2\ell_2+\cdots+\kappa\ell_\kappa=m} h^1\Big(X,\, \mathrm{Sym}^{\ell_1} T_X^* \otimes \mathrm{Sym}^{\ell_2} T_X^* \otimes \cdots \otimes \mathrm{Sym}^{\ell_\kappa} T_X^*\Big). \end{aligned}$$

Hence it is now clear that, in order to establish existence nonzero global sections of $\mathscr{E}^{GG}_{\kappa,m} T_X^*$ on hypersurfaces of general type, it would suffice that the considered sum of h^1's grows less substantially than the sum of h^0's, as κ tends to ∞ and as $m \gg \kappa$ tends to ∞ too. We conclude this section by recalling that

the Euler-Poincaré characteristic transfers better than cohomology dimensions through exact sequences, namely without inequalities.

Lemma. *The Euler-Poincaré characteristic of the Green-Griffiths bundle is equal to that of its associated graded bundle:*

$$\chi\big(X, \mathscr{E}_{\kappa,m}^{GG} T_X^*\big) = \chi\big(X, \mathsf{Gr}^\bullet \mathscr{E}_{\kappa,m}^{GG} T_X^*\big).$$

Proof. In fact, in the general context of the previous two lemmas, because for any exact sequence of finite-dimensional vector spaces:

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_N \longrightarrow 0$$

one may easily check that the alternating sum of dimensions vanishes:

$$0 = \dim A_1 - \dim A_2 + \cdots + (-1)^N \dim A_N,$$

one deduces from the long exact sequence of cohomology (6) that:

$$\begin{aligned} 0 = h^0(X,\, \mathsf{Gr}^k E) - h^0(X,\, E/E_{k+1}) + h^0(X,\, E/E_k) - \\ - h^1(X,\, \mathsf{Gr}^k E) - h^1(X,\, E/E_{k+1}) + h^1(X,\, E/E_k) + \\ + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + \\ + (-1)^n h^n(X,\, \mathsf{Gr}^k E) - (-1)^n h^n(X,\, E/E_{k+1}) + (-1)^n h^n(X,\, E/E_k), \end{aligned}$$

or else after gathering terms column by column, that:

$$0 = \chi(X,\, \mathsf{Gr}^k E) - \chi(X,\, E/E_{k+1}) + \chi(X,\, E/E_k).$$

Finally, a plain summation $\sum_{k=0}^n$ yields the formula claimed. □

§3. Euler-Poincaré characteristic computations

Theorem. ([20]) *On an arbitrary compact complex projective manifold of dimension $n \geqslant 1$, the Green-Griffiths jet bundle $\mathscr{E}_{\kappa,m}^{GG} T_X^*$ has an Euler-Poincaré characteristic asymptotically given by:*

$$\begin{aligned} \chi\big(X, \mathscr{E}_{\kappa,m}^{GG} T_X^*\big) &= \chi\big(X, \mathsf{Gr}^\bullet \mathscr{E}_{\kappa,m}^{GG} T_X^*\big) \\ &= \sum_{\ell_1 + 2\ell_2 + \cdots + \kappa\ell_\kappa = m} \chi\Big(X,\, \mathrm{Sym}^{\ell_1} T_X^* \otimes \mathrm{Sym}^{\ell_2} T_X^* \otimes \cdots \otimes \mathrm{Sym}^{\ell_\kappa} T_X^*\Big) \\ &= \frac{m^{(\kappa+1)n-1}}{((\kappa+1)n-1)!\,(\kappa!)^n} \Big\{ (\mathsf{c}_1^*)^n \frac{(\log\kappa)^n}{n!} + \mathsf{O}_n\big((\log\kappa)^{n-1}\big) \Big\} + \mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big), \end{aligned}$$

where $\mathsf{c}_1^* = \mathsf{c}_1(T_X^*) = -\mathsf{c}_1(T_X)$ is the first Chern class of T_X^*, a $(1,1)$-cohomology class on X, and where:

(i) *the first remainder is a linear combination of homogeneous[4] terms $(\mathsf{c}_1^*)^{\lambda_1} (\mathsf{c}_2^*)^{\lambda_2} \cdots (\mathsf{c}_n^*)^{\lambda_n}$ with $\lambda_1 + 2\,\lambda_2 + \cdots + n\,\lambda_n = n$, with rational coefficients all bounded in absolute value by* $\mathsf{Constant}_n\, (\log\kappa)^{n-1}$*;*

[4] As usual, we understand implicitly that each (n,n)-cohomology class $(\mathsf{c}_1^*)^{\lambda_1} \cdots (\mathsf{c}_n^*)^{\lambda_n}$ is *integrated* over X, hence represents the *numerical* value $\int_X (\mathsf{c}_1^*)^{\lambda_1} \cdots (\mathsf{c}_n^*)^{\lambda_n}$.

(ii) *the second remainder is a polynomial in m of submaximal degree $\leqslant (\kappa+1)n-2$ whose coefficients are linear combinations of the same $(\mathsf{c}_1^*)^{\lambda_1}(\mathsf{c}_2^*)^{\lambda_2}\cdots(\mathsf{c}_n^*)^{\lambda_n}$ with rational coefficients all also bounded in absolute value by* $\mathsf{Constant}_{n,\kappa}$.

In the case where $X\subset\mathbb{P}^{n+1}(\mathbb{C})$ is a hypersurface of degree $d\geqslant 1$, each homogeneous monomial $(\mathsf{c}_1^*)^{\lambda_1}(\mathsf{c}_2^*)^{\lambda_2}\cdots(\mathsf{c}_n^*)^{\lambda_n}$ — implicitly integrated on X, as usually understood in complex algebraic geometry — expresses in terms of n and d by means of some universal formulas as follows. Let $h:=\mathsf{c}_1\big(\mathscr{O}_{\mathbb{P}^{n+1}}(1)\big)$ denotes the hyperplane $(1,1)$-cohomology class, which satisfies $h^n=\int_X h^n=d$. Then one may represent ([17], p. 170):

$$\tag{8}\left[\begin{aligned}
&\mathsf{c}_1=-h\big(d-n-2\big)\\
&\mathsf{c}_2=h^2\big(d^2-\tfrac{(n+2)!}{(n+1)!\,1!}\,d+\tfrac{(n+2)!}{n!\,2!}\big)\\
&\mathsf{c}_3=-h^3\big(d^3-\tfrac{(n+2)!}{(n+1)!\,1!}\,d^2+\tfrac{(n+2)!}{n!\,2!}\,d-\tfrac{(n+2)!}{(n-1)!\,3!}\big)\\
&\cdot\cdot\ \ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\mathsf{c}_n=(-1)^n\,h^n\big(d^n-\tfrac{(n+2)!}{(n+1)!\,1!}\,d^{n-1}+\cdots+(-1)^n\,\tfrac{(n+2)!}{2!\,n!}\big).
\end{aligned}\right.$$

Recall that the Chern classes of the tangent T_X and of the cotangent bundle T_X^* of the hypersurface $X\subset\mathbb{P}^{n+1}(\mathbb{C})$ are linked together by the simple relation:

$$\mathsf{c}_k^*:=\mathsf{c}_k\big(T_X^*\big)=(-1)^k\,\mathsf{c}_k\big(T_X\big)=(-1)^k\,\mathsf{c}_k,$$

for $k=1,2,\dots,n$. It follows notably for instance that:

$$(\mathsf{c}_1^*)^n=(-1)^n(-h)^n\,\big(d-n-2\big)^n=d\big(d-n-2\big)^n.$$

Generally speaking, one easily convinces oneself that each homogeneous degree n monomial $\mathsf{c}_1^{\tau_1}\mathsf{c}_2^{\tau_2}\cdots\mathsf{c}_n^{\tau_n}$ identifies with a certain polynomial:

$$\mathsf{c}_1^{\tau_1}\mathsf{c}_2^{\tau_2}\cdots\mathsf{c}_n^{\tau_n}=\sum_{k=1}^{n+1}C_k^{\tau_1,\tau_2,\dots,\tau_n}\cdot d^k$$

with respect to $d=\deg X$ having degree $\leqslant n+1$ with integer coefficients $C_k^{\tau_1,\tau_2,\dots,\tau_n}\in\mathbb{Z}$. Furthermore, the constant coefficient $C_0^{\tau_1,\tau_2,\dots,\tau_n}=0$ is zero, because the factor $h^{\tau_1+2\tau_2+\cdots+n\tau_n}=h^n=d$ is clearly always present in every $\mathsf{c}_1^{\tau_1}\mathsf{c}_2^{\tau_2}\cdots\mathsf{c}_n^{\tau_n}$.

As a result, the Euler-Poincaré characteristic of $\mathscr{E}_{\kappa,m}^{GG}T_X^*$:

$$\tfrac{m^{(\kappa+1)n-1}}{(\kappa!)^n((\kappa+1)n-1)!}\bigg\{d\,(d-n-2)^n\frac{(\log\kappa)^n}{n!}+\mathsf{O}_{n,d}\big((\log\kappa)^{n-1}\big)\bigg\}+\mathsf{O}_{n,d,\kappa}\big(m^{(\kappa+1)n-2}\big),$$

visibly tends to $+\infty$ as m and κ tend to $+\infty$ as soon as X is of general type, that it is to say, as soon as:

$$\deg X\geqslant\dim X+3.$$

Proof. With more details, we redo Green-Griffiths' proof; fundamentals may be found in [18] (pp. 50–59 plus Chap. 15), in [3] and in [21].

To begin with, introduce the *formal root decomposition*:

$$\mathsf{c}\big(T_X^*\big) = 1 + \mathsf{c}_1^* + \mathsf{c}_2^* + \cdots + \mathsf{c}_n^* = (1+\mathsf{a}_1^*)(1+\mathsf{a}_2^*)\cdots(1+\mathsf{a}_n^*)$$

of the *total Chern class*, namely of the sum $\mathsf{c}(T_X^*)$ of the c_i^* so that c_i^* is the i-th elementary symmetric function of the *Chern roots* a_j^*:

$$\mathsf{c}_i^* = \sum_{1\leqslant j_1<j_2<\cdots<j_i\leqslant n} \mathsf{a}_{j_1}^*\mathsf{a}_{j_2}^*\cdots\,\mathsf{a}_{j_i}^* \qquad\qquad {\scriptstyle (i=0,\ 1\cdots n)}.$$

Similarly, let the a_j denote the Chern roots of $\mathsf{c}(T_X) = 1 + \mathsf{c}_1 + \cdots + \mathsf{c}_n$ and let the symbol $[\ \ ]_j$ denote projection to the (j,j)-cohomology class, so that for instance $\big[1+\mathsf{c}_1^*+\cdots+\mathsf{c}_n^*\big]_j = \mathsf{c}_j^*$. To prove the theorem, we must apply the Riemann-Roch-Hirzebruch theorem [21] which states that the Euler-Poincaré characteristic:

$$\chi\big(X,\ \mathscr{E}_{\kappa,m}^{GG}T_X^*\big) \overset{def}{=} \sum_{0\leqslant q\leqslant n} (-1)^q\,\dim H^q\big(X,\ \mathscr{E}_{\kappa,m}^{GG}T_X^*\big)$$

is equal to the integral over X:

$$\chi = \int_X \big[\mathsf{ch}\big(\mathscr{E}_{\kappa,m}^{GG}T_X^*\big)\cdot\mathsf{td}(T_X)\big]_n = \int_X \sum_{j=0}^{n} \big[\mathsf{ch}\big(\mathscr{E}_{\kappa,m}^{GG}T_X^*\big)\big]_{n-j}\big[\mathsf{td}(T_X)\big]_j$$

of the (n,n)-part of the product between the *Chern character* $\mathsf{ch}\big(\mathscr{E}_{\kappa,m}^{GG}T_X^*\big)$ of $\mathscr{E}_{\kappa,m}^{GG}T_X^*$ (to be computed in a while) and the *Todd class*[5] of T_X:

$$\mathsf{td}(T_X) = \frac{\mathsf{a}_1}{1-e^{-\mathsf{a}_1}}\,\frac{\mathsf{a}_2}{1-e^{-\mathsf{a}_2}}\cdots\frac{\mathsf{a}_n}{1-e^{-\mathsf{a}_n}} \ = 1 + \tfrac{1}{2}\,\mathsf{c}_1 + \tfrac{1}{12}\big[\mathsf{c}_1^2+\mathsf{c}_2\big] + \cdots.$$

As usual in asymptotic complex algebraic geometry (cf. [12, 24, 25]), for the product: $\mathsf{chern}\cdot\mathsf{todd}$, picking cohomology classes of positive degree $\geqslant 1$ in $\mathsf{td}(T_X)$ forces to pick classes in $\mathsf{ch}\big(\mathscr{E}_{\kappa,m}^{GG}T_X^*\big)$ of bidegree $\leqslant (n-1,\,n-1)$, and then the associated m-contributions are *smaller* than the maximal possible: $m^{(\kappa+1)n-1}$. More precisely:

Lemma. *For every $j=0,1,\dots,n$, one has:*

$$\big[\mathsf{ch}\big(\mathscr{E}_{\kappa,m}^{GG}T_X^*\big)\big]_{n-j} = \mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-1-j}\big)$$

and consequently all terms $\sum_{j=1}^n$ are negligible for our purposes, whence:

$$\chi\big(X,\ \mathscr{E}_{\kappa,m}^{GG}T_X^*\big) = \int_X \big[\mathsf{ch}\big(\mathscr{E}_{\kappa,m}^{GG}T_X^*\big)\big]_n + \mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big).$$

---

[5] Of course, all terms of degree $\geqslant n+1$ in the a_j are $\equiv 0$, since the associated cohomology classes vanish, as does any form of bidegree (p,q) with $p\geqslant n+1$ or $q\geqslant n+1$.

Proof. As is known, the Chern character of the jet bundle equals that of its graded decomposition, and the Chern character is both additive on direct sums and multiplicative on tensor products, so that we can write:

$$\mathsf{ch}\big(\mathscr{E}_{\kappa,m}^{GG}T_X^*\big) = \sum_{\ell_1+2\,\ell_2+\cdots+\kappa\ell_\kappa=m} \mathsf{ch}\big(\mathrm{Sym}^{\ell_1}T_X^*\big)\,\mathsf{ch}\big(\mathrm{Sym}^{\ell_2}T_X^*\big)\cdots\mathsf{ch}\big(\mathrm{Sym}^{\ell_\kappa}T_X^*\big).$$

Furthermore, recall that the Chern character of an arbitrary symmetric power of T_X^* is given, in terms of the a_i^*, by the known formula:

$$\mathsf{ch}\big(\mathrm{Sym}^{\ell}T_X^*\big) = \sum_{\substack{x_1+\cdots+x_n=\ell\\ x_1,\dots,x_n\in\mathbb{N}}} e^{x_1\mathsf{a}_1^*+\cdots+x_n\mathsf{a}_n^*}.$$

We can therefore apply this to $\ell=\ell_\lambda$ for all λ with $1\leqslant\lambda\leqslant\kappa$:

$$\mathsf{ch}\big(\mathrm{Sym}^{\ell_\lambda}T_X^*\big) = \sum_{x_{\lambda 1}+\cdots+x_{\lambda n}=\ell_\lambda} e^{x_{\lambda 1}\,\mathsf{a}_1^*+\cdots+x_{\lambda n}\,\mathsf{a}_n^*},$$

where we have introduced nonnegative integers $x_{\lambda i}\in\mathbb{N}$, $i=1,\dots,n$ parametrized by λ. When we expand the product of all the κ sums involved, the exponentiated terms add up and the obtained sum together with the initial sum $\sum_{\ell_1+\cdots+\kappa\ell_\kappa=m}$ unify as a single big sum:

$$\mathsf{ch}\big(\mathscr{E}_{\kappa,m}^{GG}T_X^*\big) = \sum_{\substack{x_{11}+\cdots+x_{1n}\\ +\cdots\cdots\cdots\cdots\cdots+\\ +\kappa(x_{\kappa 1}+\cdots+x_{\kappa n})=m}} \exp\big\{(x_{11}+\cdots+x_{\kappa 1})\mathsf{a}_1^*+\cdots+(x_{1n}+\cdots+x_{\kappa n})\mathsf{a}_n^*\big\}$$

in which the ℓ_λ have been naturally removed, with the only constraint that $\sum_{\lambda=1}^{\kappa}\lambda\,(x_{\lambda 1}+\cdots+x_{\lambda n})$ be constant equal to m. Now we observe the general summation rule:

$$\textstyle\sum_{u_1+u_2+\cdots+u_\mu=m}\equiv\sum_{u_2+\cdots+u_\mu\leqslant m},$$

by simply taking $u_1:=m-u_2-\cdots-u_\mu$, where the $u_j\in\mathbb{N}$. Thus, we may eliminate x_{11} in our argument of summation and it follows at once for any $j=0,1,\dots,n$ that the quantity we want to estimate is equal to:

$$\big[\mathsf{ch}\big(\mathscr{E}_{\kappa,m}^{GG}T_X^*\big)\big]_{n-j} = \sum_{\substack{x_{12}+\cdots+x_{1n}\\ +\cdots\cdots\cdots\cdots\cdots+\\ +\kappa(x_{\kappa 1}+\cdots+x_{\kappa n})\leqslant m}} \frac{1}{(n-j)!}\,\big[(\widehat{x_{11}}+x_{21}+\cdots+x_{\kappa 1})\mathsf{a}_1^*+\cdots+(x_{1n}+\cdots+x_{\kappa n})\mathsf{a}_n^*\big]^{n-j},$$

where the symbol $\widehat{x_{11}}$ means that x_{11} is replaced by its value $m-x_{12}-\cdots-\kappa\,x_{\kappa n}$. Classically, by making the change of variables:

$$y_{12}:=\frac{x_{12}}{m},\ \dots,\ y_{1n}:=\frac{x_{1n}}{m},\ \dots\dots,\ y_{\kappa 1}:=\frac{x_{\kappa 1}}{m},\ \dots,\ y_{\kappa n}:=\frac{x_{\kappa n}}{m},$$

the discrete Riemann-like sum just obtained can be approximated by a continuous integral performed on a $(\kappa n-1)$-dimensional simplex against the standard

measure of $\mathbb{R}_+^{\kappa n-1}$:

$$\big[\mathsf{ch}\big(\mathscr{E}_{\kappa,m}^{GG}T_X^*\big)\big]_{n-j} = m^{\kappa n-1+n-j} \int_{\substack{y_{12}+\cdots+y_{1n} \\ + \cdots\cdots\cdots\cdots + \\ +\kappa(y_{\kappa 1}+\cdots+y_{\kappa n})\leqslant 1}} dy_{12}\cdots dy_{1n}\cdots\cdots dy_{\kappa 1}\cdots dy_{\kappa n}\cdot$$
$$\cdot \frac{1}{(n-j)!}\big[(\widehat{y_{11}}+y_{21}+\cdots+y_{\kappa 1})\mathsf{a}_1^*+\cdots+(y_{1n}+\cdots+y_{\kappa n})\mathsf{a}_n^*\big]^{n-j}+$$
$$+\,\mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-j-2}\big),$$

the remainder being automatically at most of the order of the submaximal power of m. The integral remaining being visibly independent of m, the conclusion is got. □

Consequently, in order to compute asymptotically our Euler-Poincaré characteristic, we only have to estimate the integral above for $j=0$, in which $\widehat{y_{11}}$ is of course an abbreviation for $1-y_{12}-\cdots-\kappa\,y_{\kappa n}$. To this aim, we make the multidilational change of variables: $y_{\lambda i}\mapsto \lambda\,y_{\lambda i}=: z_{\lambda i}$ and the asymptotic under study becomes an integral over the *standard* $(n\kappa-1)$-dimensional simplex:

$$\chi\big(X,\,\mathscr{E}_{\kappa,m}^{GG}T_X^*\big) = \int_X \big[\mathsf{ch}(\mathscr{E}_{\kappa,m}^{GG}T_X^*)\big]_n + \mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big)$$
$$\equiv \frac{m^{(\kappa+1)n-1}}{n!\,(\kappa!)^n}\int_{\substack{z_{21}+\cdots+z_{1n} \\ + \cdots\cdots\cdots\cdots + \\ +z_{\kappa 1}+\cdots+z_{\kappa n}\leqslant 1}} dz_{12}\cdots dz_{1n}\cdots\cdots dz_{\kappa 1}\cdots dz_{\kappa n}\cdot$$
$$\cdot\left[\left(\widehat{z_{11}}+\frac{z_{21}}{2}+\cdots+\frac{z_{\kappa 1}}{\kappa}\right)\mathsf{a}_1^*+\cdots+\left(\frac{z_{1n}}{1}+\frac{z_{2n}}{2}+\cdots+\frac{z_{\kappa n}}{\kappa}\right)\mathsf{a}_n^*\right]^n,$$

where now the sign "$\equiv$" means modulo $\mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big)$ and where $\widehat{z_{11}}=1-z_{12}-\cdots-z_{\kappa n}$. Applying now Newton's multinomial formula:

$$\big(Z_1+Z_2+\cdots+Z_n\big)^n = \sum_{q_1+q_2+\cdots+q_n=n}\frac{n!}{q_1!\,q_2!\,\cdots\,q_n!}\,(Z_1)^{q_1}(Z_2)^{q_2}\cdots(Z_n)^{q_n},$$

we may expand the n-th power in the second line above, getting:

$$\chi\big(X,\,\mathscr{E}_{\kappa,m}^{GG}T_X^*\big) = \frac{m^{(\kappa+1)n-1}}{\underline{n!}_\circ\,(\kappa!)^n}\sum_{q_1+\cdots+q_n=n}\frac{\underline{n!}_\circ}{q_1!\,\cdots\,q_n!}\,(\mathsf{a}_1^*)^{q_1}\cdots(\mathsf{a}_n^*)^{q_n}\cdot$$
$$\cdot\int_{\substack{z_{21}+\cdots+z_{1n} \\ + \cdots\cdots\cdots\cdots + \\ +z_{\kappa 1}+\cdots+z_{\kappa n}\leqslant 1}} dz_{21}\cdots dz_{1n}\cdots\cdots dz_{\kappa 1}\cdots dz_{\kappa n}\cdot$$
$$\cdot\left(\widehat{z_{11}}+\frac{z_{21}}{2}+\cdots+\frac{z_{\kappa 1}}{\kappa}\right)^{q_1}\cdots\left(\frac{z_{1n}}{1}+\frac{z_{2n}}{2}+\cdots+\frac{z_{\kappa n}}{\kappa}\right)^{q_n}.$$

The $n!$ drops, a fact denoted with the symbol "$\underline{}_\circ$". Furthermore, in the integral — call it $\mathsf{I}_{q_1,\ldots,q_n}$ — which appears naturally in the last two lines, we yet

expand the q_1-th, ..., the q_n-th powers:

$$\begin{aligned}\mathsf{I}_{q_1,\dots,q_n} = \sum_{q_{11}+q_{21}+\cdots+q_{\kappa 1}=q_1} \cdots \sum_{q_{1n}+q_{2n}+\cdots+q_{\kappa n}} & \frac{q_1!}{q_{11}!\,q_{21}!\,\cdots\,q_{\kappa 1}!} \cdots \frac{q_n!}{q_{1n}!\,q_{2n}!\,\cdots\,q_{\kappa n}!}\cdot \\ & \cdot \frac{1}{(2)^{q_{21}}\,\cdots\,(\kappa)^{q_{\kappa 1}}} \cdots \frac{1}{(1)^{q_{1n}}\,(2)^{q_{2n}}\,\cdots\,(\kappa)^{q_{\kappa n}}}\cdot \\ \cdot \int_{\substack{z_{21}+\cdots+z_{1n} \\ +\cdots\cdots\cdots+ \\ +z_{\kappa 1}+\cdots+z_{\kappa n}\leqslant 1}} & dz_{21}\cdots dz_{1n}\cdots\cdots dz_{\kappa 1}\cdots dz_{\kappa n}\cdot \\ & \cdot (\widehat{z_{11}})^{q_{11}}\,(z_{21})^{q_{21}}\,\cdots\,(z_{\kappa 1})^{q_{\kappa 1}}\,\cdots\cdots\,(z_{1n})^{q_{1n}}\,(z_{2n})^{q_{2n}}\,\cdots\,z_{\kappa n}^{q_{\kappa n}}.\end{aligned}$$

Lemma. *For any integer $p \geqslant 2$ and for any nonnegative integer exponents $j_1, j_2, \dots, j_p \in \mathbb{N}$, one has:*

$$\int_{\substack{u_2+\cdots+u_p\leqslant 1 \\ u_2\geqslant 0,\ \dots,\ u_p\geqslant 0}} [1-u_2-\cdots-u_p]^{j_1}\,u_2^{j_2}\cdots u_p^{j_p}\,du_2\cdots du_p = \frac{j_1!\ j_2!\ \cdots\ j_p!}{(j_1+j_2+\cdots+j_p+p-1)!}.$$

Proof. By decomposing the integrations, we may write this integral as:

$$\int_0^1 u_2^{j_2}\,du_2 \int_0^{1-u_2} u_3^{j_3}\,du_3\ \cdots\cdots \int_0^{1-u_2-\cdots-u_{p-1}} (1-u_2-\cdots-u_{p-1}-u_p)^{j_1}\,u_p^{j_p}\,du_p =: \mathsf{J}^p_{j_1,j_2,j_3,\dots,j_p}.$$

Taking j_p+1 times the primitive of the first factor in the last integral and integrating successively by parts, this integral in question receives the value:

$$\left[-\frac{(1-u_2-\cdots-u_{p-1}-u_p)^{j_1+j_p+1}}{(j_1+1)\cdots(j_1+j_p)(j_1+j_p+1)}\,j_p!\right]_0^{1-u_2-\cdots-u_{p-1}} = \frac{j_1!}{(j_1+j_p+1)!}\,(1-u_2-\cdots-u_{p-1})^{j_1+j_p+1}\,j_p!.$$

Thus, the case $p=2$ is settled. If $p\geqslant 3$, inserting this value just computed:

$$\mathsf{J}^p_{j_1,j_2,\dots,j_{p-1},j_p} = \frac{j_1!\,j_p!}{(j_1+j_p+1)!}\,\mathsf{J}^{p-1}_{j_1+j_p+1,j_2,\dots,j_{p-1}} = \frac{j_1!\,j_p!}{\underline{(j_1+j_p+1)!}_\circ}\,\frac{\underline{(j_1+j_p+1)!}_\circ\,j_2!\cdots j_{p-1}!}{(j_1+j_p+1+j_2+\cdots+j_{p-1}+p-2)!},$$

we get without effort the general conclusion by induction on p. □

So applying this elementary lemma, we may finish to compute our integral:

$$\begin{aligned}\mathsf{I}_{q_1,\dots,q_n} = \sum_{q_{11}+q_{21}+\cdots+q_{\kappa 1}=q_1} \cdots \sum_{q_{1n}+q_{2n}+\cdots+q_{\kappa n}} & \frac{q_1!}{\underline{q_{11}!\,q_{21}!\,\cdots\,q_{\kappa 1}!}_\circ} \cdots \frac{q_n!}{\underline{q_{1n}!\,q_{2n}!\,\cdots\,q_{\kappa n}!}_\circ}\cdot \\ & \cdot \frac{1}{(2)^{q_{21}}!\,\cdots\,(\kappa)^{q_{\kappa 1}}} \cdots \frac{1}{(1)^{q_{1n}}\,(2)^{q_{2n}}\,\cdots\,(\kappa)^{q_{\kappa n}}}\cdot \\ & \cdot \frac{\underline{q_{11}!\,q_{21}\,\cdots\,q_{\kappa 1}!}_\circ\ \cdots\cdots\ \underline{q_{1n}!\,q_{2n}!\,\cdots\,q_{\kappa n}!}_\circ}{(q_{11}+q_{21}+\cdots+q_{\kappa 1}+\cdots\cdots+q_{1n}+q_{2n}+\cdots+q_{\kappa n}+\kappa n-1)!}.\end{aligned}$$

Remarkably, all the factorials $q_{\lambda i}!$ drop. Furthermore, the big factorial in the denominator visibly simplifies as

$$(q_1+\cdots+q_n+\kappa n-1)! = (n+\kappa n-1)!,$$

and we get a formula for $\mathsf{I}_{q_1,\dots,q_n}$ in which it will appear soon to be convenient to reconstitute a product of n independent big sums, and to this aim, we add in

advance the innocuous factor $\frac{1}{(1)^{q_{11}}}$:

$$
\begin{aligned}
&\mathsf{I}_{q_1,\dots,q_n} = \\
&= \frac{q_1!\cdots q_n!}{(q_1+\cdots+q_n+\kappa n-1)!}\sum_{q_{11}+\cdots+q_{\kappa 1}=q_1}\cdots\sum_{q_{1n}+\cdots+q_{\kappa n}=q_n}\frac{1}{(1)^{q_{11}}\cdots(\kappa)^{q_{\kappa 1}}}\cdots\frac{1}{(1)^{q_{1n}}\cdots(\kappa)^{q_{\kappa n}}}\\
&= \frac{q_1!\cdots q_n!}{((\kappa+1)n-1)!}\Bigg(\sum_{q_{11}+\cdots+q_{\kappa 1}=q_1}\frac{1}{(1)^{q_{11}}\cdots(\kappa)^{q_{\kappa 1}}}\Bigg)\cdots\Bigg(\sum_{q_{1n}+\cdots+q_{\kappa n}=q_n}\frac{1}{(1)^{q_{1n}}\cdots(\kappa)^{q_{\kappa n}}}\Bigg).
\end{aligned}
$$

Now, when we plug this formula in the computation of $\chi\big(X,\ \mathscr{E}_{\kappa,m}^{GG}T_X^*\big)$ that we interrupted before stating the lemma, all the factorials $q_1!,\dots,q_n!$ appear once at a numerator place and once at a denominator place, so they drop all and we finally get:

$$
\begin{aligned}
\chi\big(X,\ \mathscr{E}_{\kappa,m}^{GG}T_X^*\big) = \frac{m^{(\kappa+1)n-1}}{(\kappa!)^n\,((\kappa+1)n-1)!}\Bigg[&\sum_{q_1+\cdots+q_n=n}(\mathsf{a}_1^*)^{q_1}\cdots(\mathsf{a}_n^*)^{q_n}\cdot\\
&\cdot\Bigg(\sum_{q_{11}+\cdots+q_{\kappa 1}=q_1}\frac{1}{(1)^{q_{11}}\cdots(\kappa)^{q_{\kappa 1}}}\Bigg)\cdots\Bigg(\sum_{q_{1n}+\cdots+q_{\kappa n}=q_n}\frac{1}{(1)^{q_{1n}}\cdots(\kappa)^{q_{\kappa n}}}\Bigg)\Bigg]+\\
&+\mathsf{O}\big(m^{(\kappa+1)n-2}\big).
\end{aligned}
$$

We therefore have to deal with the asymptotic character, as $\kappa\to\infty$, of the polylogarithmic sums of the type:

$$
\Sigma_1^\kappa(q) := \sum_{\substack{q_1+\cdots+q_\kappa=q\\ q_1\geqslant 0,\ \cdots,\ q_\kappa\geqslant 0}}\frac{1}{(1)^{q_1}\cdots(\kappa)^{q_\kappa}},
$$

where $q\in\mathbb{N}$ is arbitrary.

Lemma. *As $\kappa\to\infty$, one has:*

$$
\Sigma_1^\kappa(q) = \frac{(\log\kappa)^q}{q!}+\mathsf{O}_n\big((\log\kappa)^{q-1}\big).
$$

Proof. Easily re-doable, and in fact also known in the literature on polylogarithms ([9]). □

From this last lemma, it follows at once that:

$$
\begin{aligned}
\chi\big(X,\ \mathscr{E}_{\kappa,m}^{GG}T_X^*\big) &= \frac{m^{(\kappa+1)n-1}}{((\kappa+1)n-1)!\,(\kappa!)^n}\Bigg[\sum_{q_1+\cdots+q_n=n}(\mathsf{a}_1^*)^{q_1}\cdots(\mathsf{a}_n^*)^{q_n}\cdot\\
&\qquad\cdot\frac{(\log\kappa)^{q_1}}{q_1!}\cdots\frac{(\log\kappa)^{q_n}}{q_n!}+\mathsf{O}_n\big((\log\kappa)^{n-1}\big)\Bigg]+\mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big)\\
&= \tfrac{m^{(\kappa+1)n-1}}{(\kappa!)^n\,((\kappa+1)n-1)!}\Big[(\mathsf{a}_1^*+\cdots+\mathsf{a}_n^*)^n\,\tfrac{(\log\kappa)^n}{n!}+\mathsf{O}_n\big((\log\kappa)^{n-1}\big)\Big]+\mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big),
\end{aligned}
$$

so the asymptotic formula exhibited in the theorem is established. To conclude the proof, one easily convinces oneself by inspecting the remainders that they indeed have the form claimed in **(i)** and **(ii)**. □

Open problem. Applying the concepts and the combinatorics partly achieved in [9, 4, 47], find *closed explicit formulas* firstly for the remainder terms

$\mathsf{O}_n\big((\log\kappa)^{n-1}\big)$, secondly, for the remainder terms $\mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big)$. As an accessible preliminary, study the $\Sigma_q(\kappa)$ completely.

§4. Exact Schur Bundle Decomposition

Schur bundles and Pieri rule. Thanks to the filtration provided by the theorem on p. 10 and to the basic cohomology inequalities reproved in Section 2, the study of the Green-Griffiths jet bundle can in principle be led back to the study of multitensored symmetric powers:

$$\mathrm{Sym}^{\ell_1} T_X^* \otimes \mathrm{Sym}^{\ell_2} T_X^* \otimes \cdots \otimes \mathrm{Sym}^{\ell_\kappa} T_X^*$$

of the cotangent bundle. But it is known since the works of Isai Schur at the turn to the 20th century that these multitensored bundles can even be decomposed in more atomic independent bricks.

Since the complex linear group $\mathsf{GL}_n(\mathbb{C})$ acts naturally on T_X^* and on all of its tensor powers $\big(T_X^*\big)^{\otimes r}$ as well ($r = 1, 2, 3, \dots$), then by fundamental facts of representation theory (Schur's theorems), it follows that the (in fact complicated) direct sum $\mathsf{Gr}^\bullet \mathscr{E}_{\kappa,m}^{GG} T_X^*$ provided by the theorem on p. 10 can in principle be represented as a certain direct sum of the so-called *Schur bundles*:

$$\mathscr{S}^{(\ell_1,\ell_2,\dots,\ell_n)} T_X^*,$$

in which $\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n \geqslant 0$; we employ the notation of [36] and the reader is referred to the works of Brückmann [5, 7, 6] and to the monographs [19, 45, 26, 22, 34] for background material, or alternatively to p. 70 below. In order to determine which $\mathscr{S}^{(\ell_1,\dots,\ell_n)} T_X^*$ appear in $\mathsf{Gr}^\bullet \mathscr{E}_{\kappa,m}^{GG} T_X^*$, possibly with some multiplicity $\geqslant 1$, two options present themselves.

The first option would be to apply step by step the so-called *Pieri formula* ([19], p. 455) to the direct sum representation:

$$\mathsf{Gr}^\bullet \mathscr{E}_{\kappa,m}^{GG} T_X^* = \bigoplus_{\ell_1+2\ell_2+\cdots+\kappa\ell_\kappa = m} \mathscr{S}^{(\ell_1,0,\dots,0)} T_X^* \otimes \mathscr{S}^{(\ell_2,0,\dots,0)} T_X^* \otimes \cdots \otimes \mathscr{S}^{(\ell_\kappa,0,\dots,0)} T_X^*.$$

Pieri indeed provides a neat combinatorial rule for representing any tensor product of a Schur bundle with a symmetric power as a certain direct sum of well controlled Schur bundles over X:

$$\mathscr{S}^{(t_1,\dots,t_n)} T_X^* \otimes \mathscr{S}^{(\ell,0,\dots,0)} T_X^* = \sum_{\substack{s_1+\cdots+s_n = \ell+t_1+\cdots+t_n \\ s_1 \geqslant t_1 \geqslant s_2 \geqslant t_2 \geqslant \cdots \geqslant s_n \geqslant t_n \geqslant 0}} \mathscr{S}^{(s_1,\dots,s_n)} T_X^*. \tag{9}$$

However, when one tries to induct on such a formula, the complexity increases dramatically as soon the number κ of tensor factors in $\mathsf{Gr}^\bullet \mathscr{E}_{\kappa,m}^{GG} T_X^*$ passes above $\kappa = 5$, even in dimension $n = 2$, and apparently, nothing really effective or exploitable for us exists in the literature.

Invariant theory approach. The second option, more direct and more suited to asymptotic approximations, consists in interpreting the problem directly in terms of classical invariant theory, starting with the original definition (3) on p. 6 of Green-Griffiths jets. Indeed, the general $n \times n$ complex unipotent matrix:

$$
\mathsf{u} := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \mathsf{u}_{21} & 1 & 0 & \cdots & 0 \\ \mathsf{u}_{31} & \mathsf{u}_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \mathsf{u}_{n1} & \mathsf{u}_{n2} & \mathsf{u}_{n3} & \cdots & 1 \end{pmatrix},
$$

where the $\mathsf{u}_{ij} \in \mathbb{C}$ are arbitrary complex numbers, acts naturally and linearly on all the jet variables in such a way that for any jet level λ with $1 \leqslant \lambda \leqslant \kappa$, one sets in matrix notation:

$$
g^{(\lambda)} := \mathsf{u} \cdot f^{(\lambda)},
$$

that is to say in greater length:

$$
\begin{cases} g_1^{(\lambda)} := f_1^{(\lambda)} \\ g_2^{(\lambda)} := f_2^{(\lambda)} + \mathsf{u}_{21}\, f_1^{(\lambda)} \\ g_3^{(\lambda)} := f_3^{(\lambda)} + \mathsf{u}_{32}\, f_2^{(\lambda)} + \mathsf{u}_{31}\, f_1^{(\lambda)} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ g_n^{(\lambda)} = f_n^{(\lambda)} + \mathsf{u}_{n,n-1}\, f_{n-1}^{(\lambda)} + \cdots + \mathsf{u}_{n1}\, f_1^{(\lambda)}. \end{cases}
$$

A general fact from the classical representation theory of $\mathsf{GL}_n(\mathbb{C})$ states that the so-called *vectors of highest weight* identify precisely to those that remain invariant by this unipotent action, namely to jet polynomials $\mathsf{P}\big(j^\kappa f\big)$ which satisfy the invariancy condition:

$$
\mathsf{P}\big(j^\kappa g\big) = \mathsf{P}\big(\mathsf{u} \cdot j^\kappa f\big) \equiv \mathsf{P}\big(j^\kappa f\big),
$$

for every unipotent matrix $\mathsf{u} \in \mathsf{U}_n(\mathbb{C})$. Furthermore and most importantly, there is a one-to-one correspondence between the vectors of highest weight and the Schur bundles appearing in the decomposition of $\mathsf{Gr}^\bullet \mathscr{E}_{\kappa,m}^{GG} T_X^*$, the rule being as follows. Precisely speaking, the vector space of unipotent-invariant polynomials (vectors of highest weight) is shown to decompose as a direct sum of (linearly independent) one-dimensional spaces generated by vectors $\mathsf{Q} = \mathsf{Q}\big(j^\kappa f\big)$ that are eigenvalues for the action $\mathsf{e} \cdot f_i^{(\lambda)} := \mathsf{e}_i\, f_i^{(\lambda)}$ of all diagonal matrices of the form:

$$
\mathsf{e} := \begin{pmatrix} \mathsf{e}_1 & 0 & \cdots & 0 \\ 0 & \mathsf{e}_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \mathsf{e}_n \end{pmatrix},
$$

where $\mathsf{e}_1, \mathsf{e}_2, \dots, \mathsf{e}_n$ are arbitrary complex numbers, so that there are certain characteristic exponents ℓ_i with the property that:

$$\mathsf{Q}\big(\mathsf{e} \cdot j^\kappa f\big) = (\mathsf{e}_1)^{\ell_1} (\mathsf{e}_2)^{\ell_2} \cdots (\mathsf{e}_n)^{\ell_n}\, \mathsf{Q}\big(j^\kappa f\big).$$

One shows that $\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n \geqslant 0$ and that such an eigenvector Q (of highest weight) is precisely linked to the Schur bundle $\mathscr{S}^{(\ell_1,\ell_2,\dots,\ell_n)} T_X^*$ which corresponds to an irreducible representation on a fiber over a point $x \in X$. Of course, a specific Schur bundle $\mathscr{S}^{(\ell_1,\ell_2,\dots,\ell_n)} T_X^*$ could well occur several times in the sought decomposition of $\mathsf{Gr}^\bullet \mathscr{E}_{\kappa,m}^{GG} T_X^*$, hence have a certain multiplicity $\geqslant 2$, because some different linearly independent Q's could share the same characteristic exponents ℓ_i. In fact, this will indeed be the case below, and determining such multiplicities, at least asymptotically as $\kappa \to \infty$, will be crucial for us.

Serendipity. The knowledge of the algebra of invariants of the full unipotent group $\mathsf{U}_n(\mathbb{C}) \subset \mathsf{GL}_n(\mathbb{C})$ dates back to the nineteenth century. As a matter of fact, the following four basic statements Theorems A, B, C and D below, which will precede a main starting theorem specially designed for our future purposes, are essentially known and they are established in various sources.

Theorem A. ([45, 22, 31, 34]) *The algebra of jet polynomials invariant under the above action of the full unipotent group* $\mathsf{U}_n(\mathbb{C}) \subset \mathsf{GL}_n(\mathbb{C})$ *is* generated, as an algebra*, by the collection of all the determinants (minors):*

$$\left|f_1^{(\lambda_1)}\right| =: \Delta_1^{\lambda_1}, \quad \begin{vmatrix} f_1^{(\lambda_1)} & f_2^{(\lambda_1)} \\ f_1^{(\lambda_2)} & f_2^{(\lambda_2)} \end{vmatrix} =: \Delta_{1,2}^{\lambda_1,\lambda_2}, \quad \begin{vmatrix} f_1^{(\lambda_1)} & f_2^{(\lambda_1)} & f_3^{(\lambda_1)} \\ f_1^{(\lambda_2)} & f_2^{(\lambda_2)} & f_3^{(\lambda_2)} \\ f_1^{(\lambda_3)} & f_2^{(\lambda_3)} & f_3^{(\lambda_3)} \end{vmatrix} =: \Delta_{1,2,3}^{\lambda_1,\lambda_2,\lambda_3},$$

$$\dots\dots, \quad \begin{vmatrix} f_1^{(\lambda_1)} & f_2^{(\lambda_1)} & \cdots & f_n^{(\lambda_1)} \\ f_1^{(\lambda_2)} & f_2^{(\lambda_2)} & \cdots & f_n^{(\lambda_2)} \\ \cdots & \cdots & \cdots & \cdots \\ f_1^{(\lambda_n)} & f_2^{(\lambda_n)} & \cdots & f_n^{(\lambda_n)} \end{vmatrix} =: \Delta_{1,2,\dots,n}^{\lambda_1,\lambda_2,\dots,\lambda_n},$$

in which the jet orders $1 \leqslant \lambda_1 < \lambda_2 < \cdots < \lambda_n \leqslant \kappa$ *are all arbitrary* [6]*, and these determinants are visibly invariant with respect to the* $\mathsf{U}_n(\mathbb{C})$*-action.*

However, although all the determinants in question happen to be linearly independent, one cannot just pretend that the whole unipotent-invariant algebra identifies with the plain polynomial algebra:

$$\mathbb{C}\big[\Delta_1^{\lambda_1}, \Delta_{1,2}^{\lambda_1,\lambda_2}, \dots, \Delta_{1,2,\dots,n}^{\lambda_1,\dots,\lambda_n}\big],$$

[6] It is only necessary to consider strictly increasing integers λ_l, since for every i with $1 \leqslant i \leqslant n$, and for every permutation σ of $\{1, 2, \dots, i\}$ one clearly has:

$$\Delta_{1,2,\dots,i}^{\lambda_{\sigma(1)},\lambda_{\sigma(2)},\dots,\lambda_{\sigma(i)}} = (-1)^{\mathrm{sign}(\sigma)}\, \Delta_{1,2,\dots,i}^{\lambda_1,\lambda_2,\dots,\lambda_i}.$$

because several elementary *non*linear relations exist between all these determinants; for instance, there exist the basic quadratic Plücker relations[7] of first and of second type:

$$0 \equiv \Delta_1^{\lambda_1}\,\Delta_{1,2}^{\lambda_2,\lambda_3} + \Delta_1^{\lambda_3}\,\Delta_{1,2}^{\lambda_1,\lambda_2} + \Delta_1^{\lambda_2}\,\Delta_{1,2}^{\lambda_3,\lambda_1}$$
$$0 \equiv \Delta_{1,2}^{\lambda_1,\lambda_2}\,\Delta_{1,2}^{\lambda_3,\lambda_4} + \Delta_{1,2}^{\lambda_1,\lambda_4}\,\Delta_{1,2}^{\lambda_2,\lambda_3} + \Delta_{1,2}^{\lambda_1,\lambda_3}\,\Delta_{1,2}^{\lambda_4,\lambda_2},$$

so that the binomial in the first line $\Delta_1^{\lambda_1}\Delta_{1,2}^{\lambda_2,\lambda_3}$, viewed in the plain polynomial algebra $\mathbb{C}\big[\Delta_1^{\lambda_1},\,\Delta_{1,2}^{\lambda_1,\lambda_2}\big]$, would possess *two* distinct representations: itself, and:

$$-\,\Delta_1^{\lambda_3}\,\Delta_{1,2}^{\lambda_1,\lambda_2} - \Delta_1^{\lambda_2}\,\Delta_{1,2}^{\lambda_3,\lambda_1}.$$

Fortunately, the ideal of all relations between these Δ-determinants is also completely known and understood. However, presenting *explicitly* this ideal of all relations requires a bit of preparation and a few more indices.

Ideal of relations between all Δ jet-determinants. We therefore consider the collection of all determinants $\Delta_{1,2,\dots,i}^{\lambda_1,\lambda_2,\dots,\lambda_i}$ for every $i \in \{1,\dots,n\}$ and for every choice of i jet-line indices $\lambda_1,\lambda_2,\dots,\lambda_i \in \{1,\dots,\kappa\}$. At first, we equip this collection with a *partial* order by declaring that:

$$\Delta_{1,2,\dots,i}^{\lambda_1,\lambda_2,\dots,\lambda_i} <_{one} \Delta_{1,2,\dots,j}^{\mu_1,\mu_2,\dots,\mu_j}$$

if firstly:

$$i \geqslant j$$

and if secondly all the following inequalities hold:

$$\lambda_1 \leqslant \mu_1, \qquad \lambda_2 \leqslant \mu_2, \qquad \dots\dots, \qquad \lambda_j \leqslant \mu_j. \tag{10}$$

Not all determinants are comparable for this order, *e.g.* $\Delta_{1,2}^{1,4}$ and $\Delta_{1,2}^{2,3}$ are *in*comparable, and similarly, $\Delta_{1,2}^{1,4}$ and $\Delta_{1,2,3}^{2,3,4}$ are incomparable too. We will now see that there is a one-to-one correspondence between incomparable Δ-determinants and (generalized) Plücker relations.

Thus, let us pick any two general determinants $\Delta_{1,\dots,i}^{\lambda_1,\dots,\lambda_i}$ and $\Delta_{1,\dots,j}^{\mu_1,\dots,\mu_j}$ that are incomparable and distinct. Permuting the pair if necessary, we may assume that $i \geqslant j$. Furthermore, if $i = j$, we may also assume without loss of generality that $(\lambda_1,\dots,\lambda_i)$ is smaller than $(\mu_1,\dots,\mu_{i=j})$ in the lexicographic ordering, namely there exists an index $s \in \{1,\dots,i=j\}$ such that:

$$\lambda_1 = \mu_1, \qquad \dots\dots, \qquad \lambda_{s-1} = \mu_{s-1}, \qquad \lambda_s < \mu_s.$$

Therefore in both cases $i > j$ and $i = j$, we at least insure by these preliminary choices that:

$$\Delta_{1,\dots,i}^{\lambda_1,\dots,\lambda_i} \not>_{one} \Delta_{1,\dots,j}^{\mu_1,\dots,\mu_j}.$$

[7] — the knowledge of which surely goes back to the seventeenth century theory, at a time when elimination was the main tool in the search for solving algebraic equations of degrees 2, 3, 4 and 5.

Since by assumption, these two determinants are incomparable, the reverse inequality must also fail:

$$\Delta_{1,\dots,i}^{\lambda_1,\dots,\lambda_i} \not<_{one} \Delta_{1,\dots,j}^{\mu_1,\dots,\mu_j},$$

and hence in the two cases $i > j$ and $i = j$, there must exist a smallest index $t \in \{1, \dots, j\}$ such that:

$$\lambda_1 \leqslant \mu_1, \quad \dots\dots, \quad \lambda_{t-1} \leqslant \mu_{t-1}, \quad \lambda_t > \mu_t,$$

because if otherwise all the inequalities (10) would hold, one would have $\Delta_{1,\dots,i}^{\lambda_1,\dots,\lambda_i} <_{one} \Delta_{1,\dots,j}^{\mu_1,\dots,\mu_j}$. In the case $i = j$, it is clear that t can only be $\geqslant s+1$.

Remind that in any circumstance, the jet-line indices of the determinants are strictly increasing:

$$\lambda_1 < \cdots < \lambda_t < \cdots < \lambda_i \qquad \text{and} \qquad \mu_1 < \cdots < \mu_t < \cdots < \mu_j.$$

Diagrammatically, we may then represent a set of inequalities with a pivotal solder, at the index t, between the μ_i and the λ_i:

$$\mu_1 < \cdots < \mu_t \underset{\mathsf{solder}}{<} \lambda_t < \cdots < \lambda_i,$$

by exhibiting, in two adjusted lines, the vertical spot where the join takes place:

$$\begin{aligned} &\mu_1 < \mu_2 < \cdots < \mu_t \quad < \underline{\mu_{t+1} < \mu_{t+2} < \cdots < \mu_j}_{\,\mathsf{FIX}} \\ &\qquad \underline{\lambda_1 < \cdots < \lambda_{t-1}}_{\,\mathsf{FIX}} < \lambda_t < \lambda_{t+1} < \cdots < \lambda_{j-1} < \lambda_j < \cdots < \lambda_i. \end{aligned}$$

Letting now $\pi \in \mathfrak{S}_{i+1}$ be any permutation of the set $\{1, 2, \dots, i, i+1\}$ with $i+1$ elements, we shall let it act on the $i+1$ elements that are *not* underlined, so that π transforms the $i+1$ integers:

$$\begin{aligned} &\mu_1 < \mu_2 < \cdots < \mu_t < \\ &\qquad\qquad < \lambda_t < \lambda_{t+1} < \cdots < \lambda_{j-1} < \lambda_j < \cdots < \lambda_i \end{aligned}$$

to the $i+1$ permuted integers (not anymore necessarily ordered increasingly):

$$\begin{aligned} &\big(\pi(\mu_1),\, \pi(\mu_2),\, \cdots,\, \pi(\mu_t), \\ &\qquad\qquad \pi(\lambda_t),\, \pi(\lambda_{t+1}),\, \dots,\, \pi(\lambda_{j-1}),\, \pi(\lambda_j),\, \dots,\, \pi(\lambda_i)\big). \end{aligned}$$

Since our Δ-determinants are skew-symmetric with respect to any permutation of their lines, it is convenient to restrict attention only to those permutations that respect strict ordering in the two blocks:

$$\begin{aligned} &\pi(\mu_1) < \pi(\mu_2) < \cdots < \pi(\mu_t) \\ \text{and:}\quad &\pi(\lambda_t) < \pi(\lambda_{t+1}) < \cdots < \pi(\lambda_{j-1}) < \pi(\lambda_j) < \cdots < \pi(\lambda_i). \end{aligned}$$

At last, we are in a position to write down the most general quadratic Plücker relations that are fundamental for the subject.

Theorem B. ([45, 22, 31, 34]) *For any two determinants $\Delta_{1,\dots,i}^{\lambda_1,\dots,\lambda_i}$ and $\Delta_{1,\dots,j}^{\mu_1,\dots,\mu_j}$ with $i \geqslant j$ that are incomparable with respect to the partial ordering "$<_{one}$", namely which have the concrete properties that:*

- *when $i > j$, there exists an index $t \in \{1, \dots, j\}$ such that:*

$$\lambda_1 \leqslant \mu_1, \quad \dots\dots, \quad \lambda_{t-1} \leqslant \mu_{t-1}, \quad \textit{but:} \quad \lambda_t > \mu_t;$$

- *when $i = j$, there exist two indices $s \in \{1, \dots, j\}$ and $t \in \{1, \dots, j\}$ with $t \geqslant s+1$ such that:*

$$\lambda_1 = \mu_1, \ \dots, \ \lambda_{s-1} = \mu_{s-1}, \ \lambda_s < \mu_s, \\ \lambda_{s+1} \leqslant \mu_{s+1}, \ \dots, \ \lambda_{t-1} \leqslant \mu_{t-1}, \ \textit{but again:} \ \lambda_t > \mu_t;$$

the following general quadratic (Plücker) relation holds identically in the ground ring $\mathbb{C}\big[f'_{i_1}, f''_{i_2}, \dots, f^{(\kappa)}_{i_\kappa}\big]$:

$$\boxed{0 \equiv \sum_{\pi \in \mathfrak{S}_{i+1}} \sum_{\substack{\pi(\lambda_t) < \cdots < \pi(\lambda_i) \\ \pi(\mu_1) < \cdots < \pi(\mu_t)}} \mathsf{sign}(\pi) \cdot \Delta^{\lambda_1, \dots, \lambda_{t-1}, \pi(\lambda_t), \pi(\lambda_{t+1}), \dots, \pi(\lambda_{j-1}), \pi(\lambda_j), \dots, \pi(\lambda_i)}_{1, \dots, t-1, t, t+1, \dots, j-1, j, \dots, i} \cdot \\ \cdot \Delta^{\pi(\mu_1), \pi(\mu_2), \dots, \pi(\mu_t), \mu_{t+1}, \mu_{t+2}, \dots, \mu_j}_{1, 2, \dots, t, t+1, t+2, \dots, j}}.$$

We will not reproduce the proof here, but extract instead from the cited references the further important information that the ideal of relations between all our Δ jet-determinants:

$$\Delta^{\lambda_1, \dots, \lambda_i}_{1, \dots, i} = \begin{vmatrix} f_1^{(\lambda_1)} & \cdots & f_i^{(\lambda_1)} \\ \cdot\cdot & \cdots & \cdot\cdot \\ f_1^{(\lambda_i)} & \cdots & f_i^{(\lambda_i)} \end{vmatrix} \\ {\scriptstyle (i = 1 \cdots n \, ; \, 1 \leqslant \lambda_1 < \cdots < \lambda_i \leqslant \kappa)}$$

is generated (as an ideal) by all the above quadratic Plücker relations. Moreover, these relations written explicitly above do constitute a *Gröbner basis* for a certain term order, presented as follows.

Introduce first as many independent variables $\nabla^{\lambda_1^i, \dots, \lambda_i^i}$ as there are Δ jet-determinants and consider the ring $\mathbb{C}\big[\nabla^{\lambda_1^1}, \dots, \nabla^{\lambda_1^n, \dots, \lambda_n^n}\big]$. Totally order these variables by declaring that:

$$\nabla^{\lambda_1^i, \dots, \lambda_i^i} <_{two} \nabla^{\mu_1^j, \dots, \mu_j^j}$$

if either $i > j$ or else if $i = j$ *and* $(\lambda_1^i, \dots, \lambda_i^i)$ comes before $(\mu_1^j, \dots, \mu_{j=i}^j)$ in the lexicographic ordering, which simply means that there exists an index $s \in \{1, \dots, j\}$ such that:

$$\lambda_1^i = \mu_1^j, \quad \dots\dots, \quad \lambda_{s-1}^i = \mu_{s-1}^j, \quad \lambda_s^i < \mu_s^j.$$

This total order extend the partial order "$<_{one}$". Finally, let also "$<_{two}$" denote the *reverse lexicographic*[8] term ordering on $\mathbb{C}\big[\nabla^{\lambda_1^1}, \dots, \nabla^{\lambda_1^n, \dots, \lambda_n^n}\big]$ that is induced by this variable ordering $<_{two}$. The set of polynomials

[8] Generally, if $x_1 <_{two} \cdots <_{two} < x_n$, the reverse lexicographic (total) ordering induced on monomials says that $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is *smaller* than $x_1^{\beta_1} \cdots x_n^{\beta_n}$ if, when reading exponents from right to left, equality holds: $\alpha_n = \beta_n, \dots, \alpha_{u+1} = \beta_{u+1}$ until a first difference occurs: $\alpha_u \neq \beta_u$ for which $\alpha_u > \beta_u$ is *bigger* than β_u.

$\mathsf{R}\big(\nabla^{\lambda_1^1}, \dots, \nabla^{\lambda_1^n,\dots,\lambda_n^n}\big)$ which annihilate identically after replacement by the determinants:

$$0 \equiv \mathsf{R}\big(\Delta_1^{\lambda_1^1}, \dots, \Delta_{1,\dots,n}^{\lambda_1^n,\dots,\lambda_n^n}\big)$$

constitutes clearly an *ideal* of $\mathbb{C}\big[\nabla^{\lambda_1^1}, \dots, \nabla^{\lambda_1^n,\dots,\lambda_n^n}\big]$.

Theorem C. ([45, 22, 31, 34]) *The ideal of relations* $\mathsf{Id\text{-}rel}(\Delta)$ *between all* Δ *jet-determinants is generated by all the Plücker relations written above. Moreover, the collection of all these Plücker relations constitutes already* per se *a* Gröbner basis *for* $\mathsf{Id\text{-}rel}(\Delta)$ *under the term ordering "$<_{two}$". Finally, the products:*

$$\Delta_{1,\dots,i}^{\lambda_1,\dots,\lambda_i} \cdot \Delta_{1,\dots,j}^{\mu_1,\dots,\mu_j}$$

of all possible incomparable pairs generate the (monomial) ideal of leading monomials of elements of $\mathsf{Id\text{-}rel}(\Delta)$.

Polynomials modulo relations. Thanks to this statement, we will be able to find a basis of the $\mathbb{C}$-vector space:

$$\textsf{all } \Delta\textsf{-polynomials} \big/ \textsf{modulo their relations}.$$

This will be very useful, for we saw that basis vectors are in one-to-one correspondence with Schur bundles $\mathscr{S}^{(\ell_1,\dots,\ell_n)}T_X^*$ (see also below). The general Gröbner basis theory then tells us that this vector space is generated by all Δ-monomials that are *not* multiple of any product of incomparable pairs (leading monomials). In order to describe explicitly this quotient vector space, we need a classical combinatorial object.

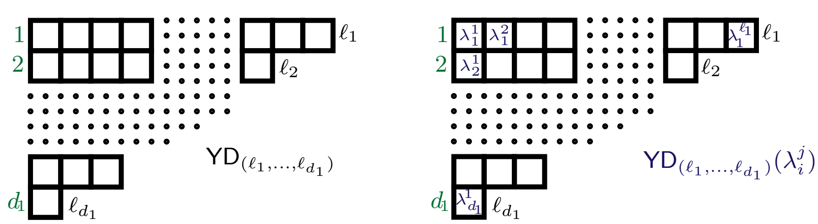
Young diagrams. Let $d_1 \geqslant 1$ be an integer and let $\ell_1, \ell_2, \dots, \ell_{d_1}$ be any collection of d_1 nonnegative integers collected in decreasing order:

$$\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_{d_1} \geqslant 1.$$

The *Young diagram* $\mathsf{YD}_{(\ell_1,\dots,\ell_{d_1})}$ associated to such a d_1-tuple $(\ell_1, \dots, \ell_{d_1})$ sits in the right-bottom quadrant $\{x \geqslant 0,\ y \leqslant 0\}$ of the plane $\mathbb{R}^2 = \mathbb{R}^2(x,y)$ and it consists, in the i-th horizontal strip $\{-i \leqslant y \leqslant -i+1\}$ from above, for $i = 1, \dots, d_1$, of the ℓ_i empty unit squares:

$$\square_i^j := \big\{(x,y) \in \mathbb{R}^2 \colon\ -i \leqslant y \leqslant -i+1,\ j-1 \leqslant x \leqslant j\big\}$$

placed, for $j = 1, \dots, \ell_i$, one after the other and starting from the vertical y-axis (left-justification).

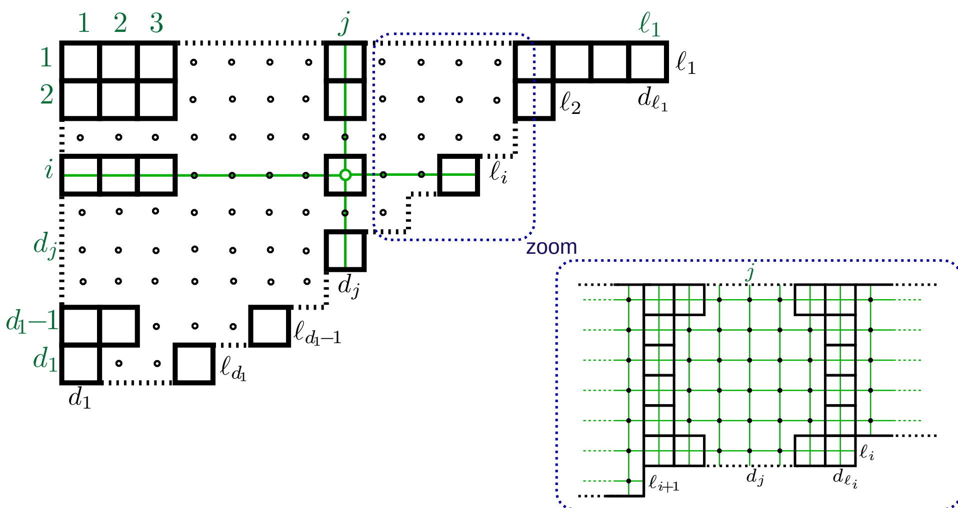


It will be convenient to give names to the column lengths, say d_j will denote that of the j-th, for $j = 1, \dots, \ell_1$. In summary and for memory:

$$\ell_i = \text{length of the } i\text{-th row}; \qquad d_j = \text{length of the } j\text{-th column}.$$

Observe that the longest column lengths are equal to d_1 for all indices j between 1 and ℓ_{d_1}, and more generally at any i-th row (see the zoom below), that the following coincidence of column lengths holds:

$$i = d_{1+\ell_{i+1}} = \cdots = d_{\ell_i} \qquad (1 \leqslant i \leqslant d_1).$$



Semi-standard Young tableaux. If $\lambda_i^j \geqslant 1$ denote as many nonnegative integers as there are empty squares $\square_i^j$, namely with $i = 1, \dots, d_1$ and $j = 1, \dots, \ell_i$, a *filling* $\mathsf{YD}_{(\ell_1, \dots, \ell_{d_1})}(\lambda_i^j)$ of the Young diagram $\mathsf{YD}_{(\ell_1, \dots, \ell_{d_1})}$ by means of the λ_i^j consists in putting each λ_i^j in each square $\square_i^j$. A *semi-standard (Young) tableau* is a filled Young diagram $\mathsf{YD}_{(\ell_1, \dots, \ell_{d_1})}(\lambda_i^j)$ having the property that when reading its full content:

$$\begin{array}{ccccccccccccc}
\lambda_1^1 & \cdots & \lambda_1^{\ell_{d_1}} & \cdots & \lambda_1^{\ell_{d_1-1}} & \cdots\;\cdots & \lambda_1^{\ell_i} & \cdots\;\cdots & \lambda_1^{\ell_2} & \cdots & \lambda_1^{\ell_1} \\
\lambda_2^1 & \cdots & \lambda_2^{\ell_{d_1}} & \cdots & \lambda_2^{\ell_{d_1-1}} & \cdots\;\cdots & \lambda_2^{\ell_i} & \cdots\;\cdots & \lambda_2^{\ell_2} & & \\
\cdot & \cdots & \cdot & \cdots & \cdot & \cdots\cdots & \cdot & \cdots & & & \\
\lambda_i^1 & \cdots & \lambda_i^{\ell_{d_1}} & \cdots & \lambda_i^{\ell_{d_1-1}} & \cdots\;\cdots & \lambda_i^{\ell_i} & & & & \\
\cdot & \cdots & \cdot & \cdots & \cdot & \cdots & & & & & \\
\lambda_{d_1-1}^1 & \cdots & \lambda_{d_1-1}^{\ell_{d_1-1}} & \cdots & \lambda_{d_1-1}^{\ell_{d_1-1}} & & & & & & \\
\lambda_{d_1}^1 & \cdots & \lambda_{d_1}^{\ell_{d_1}}, & & & & & & & &
\end{array}$$

the integers λ_i^j increase from top to bottom in each column, and they are nondecreasing[9] in each row from left to right, that is to say and more precisely:

$$\begin{aligned} &\lambda_1^j < \lambda_2^j < \cdots < \lambda_{d_j}^j && (1 \leqslant j \leqslant \ell_1) \\ &\lambda_i^1 \leqslant \lambda_i^2 \leqslant \cdots \leqslant \lambda_i^{\ell_i} && (1 \leqslant i \leqslant d_1). \end{aligned}$$

Vector space basis for the algebra of Δ jet-determinants. Coming back to our algebra of determinants $\Delta_{1,2,\dots,i}^{\lambda_1,\lambda_2,\dots,\lambda_i}$, the increasing sequence of their exponents $\lambda_1 < \lambda_2 < \cdots < \lambda_i$ will sit in a column of such a Young diagram. Since the row-size i of any not identically zero minor $\Delta_{1,2,\dots,i}^{\lambda_1,\lambda_2,\dots,\lambda_i}$ must be $\leqslant n = \operatorname{rank}(T_X^*)$, we will consider in fact only semi-standard tableaux whose depth d_1 is always $\leqslant n$. Accordingly, when it happens that $d_1 < n$ we shall adopt the natural convention that:

$$\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_{d_1-1} \geqslant \ell_{d_1} > 0 = \ell_{d_1+1} = \cdots = \ell_n.$$

We are at last in a position to state the starting point theorem.

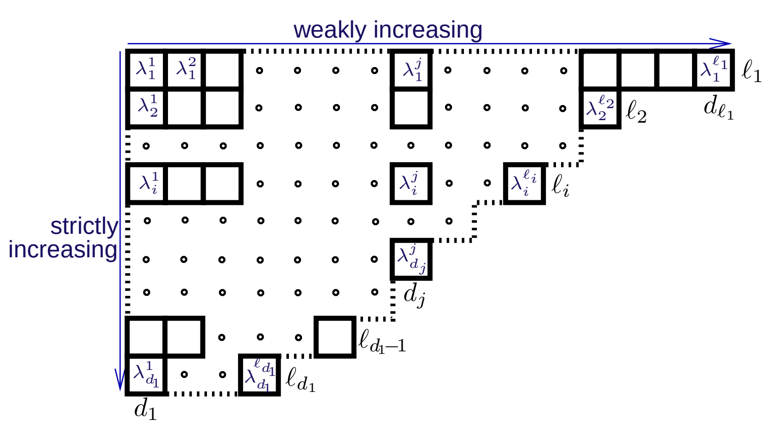
Theorem D. ([45, 22, 31, 34]) *The infinite-dimensional quotient vector space:*

$$\text{all } \Delta\text{-polynomials} \big/ \text{modulo their relations}$$

possesses a basis over $\mathbb{C}$ consisting of all possible Δ-monomials:

$$\prod_{1 \leqslant j \leqslant \ell_{d_1}} \Delta_{1,\dots,d_1}^{\lambda_1^j,\dots,\lambda_{d_1}^j} \prod_{\ell_{d_1}+1 \leqslant j \leqslant \ell_{d_1-1}} \Delta_{1,\dots,d_1-1}^{\lambda_1^j,\dots,\lambda_{d_1-1}^j} \cdots \prod_{\ell_2+1 \leqslant j \leqslant \ell_1} \Delta_1^{\lambda_1^j}$$

such that the collection of appearing upper exponents (λ_i^j) constitutes a semi-standard Young tableau:



[9] A so-called *standard tableau* would require that the integers λ_i^j also increase along the rows.

Exact Schur bundle decomposition of $\mathsf{Gr}^\bullet \mathscr{E}_{\kappa,m}^{GG} T_X^*$. In order to apply this combinatorial information to our problem, we may also represent the general Δ-monomial written above more concisely as:

$$\prod_{d_1 \geqslant i \geqslant 1} \prod_{1+\ell_{i+1} \leqslant j \leqslant \ell_i} \Delta_{1,\dots,i}^{\lambda_1^j,\dots,\lambda_i^j}. \tag{11}$$

First of all, every Δ-determinant read off from such a product happens to be an eigenvector for the action on jets of the diagonal matrices $\mathsf{e} = \mathrm{diag}\big(\mathsf{e}_1, \dots, \mathsf{e}_n\big)$:

$$\mathsf{e} \cdot \Delta_{1,2,\dots,i}^{\lambda_1^j,\lambda_2^j,\dots,\lambda_i^j} = \mathsf{e}_1\, \mathsf{e}_2 \cdots \mathsf{e}_i\, \Delta_{1,2,\dots,i}^{\lambda_1^j,\lambda_2^j,\dots,\lambda_i^j},$$

as is clear because the diagonal action just multiplies columns of such a determinant by the quantities $\mathsf{e}_1, \mathsf{e}_2, \dots, \mathsf{e}_j$:

$$\mathsf{e} \cdot \Delta_{1,2,\dots,i}^{\lambda_1^j,\lambda_2^j,\dots,\lambda_i^j} = \begin{vmatrix} \mathsf{e}_1 f_1^{(\lambda_1^j)} & \mathsf{e}_2 f_2^{(\lambda_1^j)} & \cdots & \mathsf{e}_i f_i^{(\lambda_1^j)} \\ \mathsf{e}_1 f_1^{(\lambda_2^j)} & \mathsf{e}_2 f_2^{(\lambda_2^j)} & \cdots & \mathsf{e}_i f_i^{(\lambda_2^j)} \\ \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot \\ \mathsf{e}_1 f_1^{(\lambda_i^j)} & \mathsf{e}_2 f_2^{(\lambda_i^j)} & \cdots & \mathsf{e}_i f_i^{(\lambda_i^j)} \end{vmatrix}.$$

Consequently, every general monomial in the Δ-determinants represented by the above arbitrary semi-standard tableau is also an eigenvector:

$$\begin{aligned} \mathsf{e} \cdot \bigg(\prod_{d_1 \geqslant i \geqslant 1} \prod_{1+\ell_{i+1} \leqslant j \leqslant \ell_i} \Delta_{1,\dots,i}^{\lambda_1^j,\dots,\lambda_i^j} \bigg) &= \mathsf{e} \cdot \big(\text{general } \Delta\text{-monomial}\big) \\ &= \prod_{d_1 \geqslant i \geqslant 1} \prod_{1+\ell_{i+1} \leqslant j \leqslant \ell_i} \mathsf{e}_1 \cdots \mathsf{e}_i \cdot \big(\text{same } \Delta\text{-monomial}\big) \\ &= \prod_{d_1 \geqslant i \geqslant 1} \big(\mathsf{e}_1 \cdots \mathsf{e}_i\big)^{\ell_i - \ell_{i+1}} \cdot \big(\text{same } \Delta\text{-monomial}\big) \\ &= (\mathsf{e}_1)^{\ell_1} (\mathsf{e}_2)^{\ell_2} \cdots (\mathsf{e}_n)^{\ell_n} \cdot \big(\text{same } \Delta\text{-monomial}\big). \end{aligned}$$

As a result, we deduce generally that:

$$\boxed{\text{Single semi-standard } \Delta\text{-monomial} \longleftrightarrow \text{Unique Schur bundle}},$$

and more precisely, to the general monomial associated with a semi-standard tableau $\mathsf{YD}_{(\ell_1,\dots,\lambda_n)}(\lambda_i^j)$ corresponds bijectively the Schur bundle $\mathscr{S}^{(\ell_1,\ell_2,\dots,\ell_n)} T_X^*$. Thus notably, the related Schur bundle depends only on the diagram, and it *does not depend on its filling by integers* λ_i^j.

Although essentially not new since it follows from Theorem A, B, C and D above, the following basic statement appears nowhere as such in the literature devoted the application of the jet bundle machinery to the conjectures of Green-Griffiths and of Kobayashi, but it will nonetheless constitute our basic starting point.

Theorem. *The graded vector bundle* $\mathsf{Gr}^\bullet \mathscr{E}_{\kappa,m}^{GG} T_X^*$ *associated to the bundle* $\mathscr{E}_{\kappa,m}^{GG} T_X^*$ *of* κ*-th* m*-weighted Green-Griffiths jets identifies to the following* exact direct sum *of Schur bundles:*

$$\mathsf{Gr}^\bullet \mathscr{E}_{\kappa,m}^{GG} T_X^* = \bigoplus_{\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n \geqslant 0} \Big(\mathscr{S}^{(\ell_1,\ell_2,\dots,\ell_n)} T_X^* \Big)^{\oplus M_{\ell_1,\ell_2,\dots,\ell_n}^{\kappa,m}},$$

with multiplicities $M_{\ell_1,\ell_2,\dots,\ell_n}^{\kappa,m} \in \mathbb{N}$ *equal to the number of times a Young diagram* $\mathsf{YD}_{(\ell_1,\dots,\ell_n)}$ *with row lengths equal to* $\ell_1, \ell_2, \dots, \ell_n$ *can be filled in with positive integers* $\lambda_i^j \leqslant \kappa$ *placed at its* i*-th line and* j*-th column so as to constitute a semi-standard tableau, with the further constraint that the sum of all such integers:*

$$\begin{aligned} m = \lambda_1^1 + \cdots + \lambda_1^{\ell_n} + \cdots + \lambda_1^{\ell_2} + \cdots + \lambda_1^{\ell_1} \\ + \lambda_2^1 + \cdots + \lambda_2^{\ell_n} + \cdots + \lambda_2^{\ell_2} \\ + \cdots\cdots\cdots\cdots\cdots\cdots + \\ + \lambda_n^1 + \cdots + \lambda_n^{\ell_n} \end{aligned}$$

equals the prescribed weighted homogeneity degree m*.*

This apparently complete statement should not hide the fact that the exact computation of the multiplicities $M_{\ell_1,\ell_2,\dots,\ell_n}^{\kappa,m}$ is not provided in terms of κ, m and $\ell_1, \ell_2, \dots, \ell_n$. Manual attempts to find a usable, closed and explicit formula for $M_{\ell_1,\ell_2,\dots,\ell_n}^{\kappa,m}$ showed us that the task could be hard, and we will proceed differently, in an asymptotic manner, so as to avoid several unnecessary computations which would anyway be inaccessible to us.

Corollary. *One has the following inequalities between the cohomology dimensions* h^q *for all* $q = 1, 2, \dots, n$*:*

$$\begin{aligned} h^q\big(X,\, \mathscr{E}_{\kappa,m}^{GG} T_X^*\big) &\leqslant \sum_{\ell_1 + 2\ell_2 + \cdots + \kappa\ell_\kappa = m} h^q\Big(X,\, \mathrm{Sym}^{\ell_1} T_X^* \otimes \mathrm{Sym}^{\ell_2} T_X^* \otimes \cdots \otimes \mathrm{Sym}^{\ell_\kappa} T_X^*\Big) \\ &\leqslant \sum_{\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n \geqslant 0} M_{\ell_1,\ell_2,\dots,\ell_n}^{\kappa,m} h^q\big(X,\, \mathscr{S}^{(\ell_1,\ell_2,\dots,\ell_n)} T_X^*\big). \end{aligned}$$

Proof. The first one was already derived on p. 11. Then the decomposition into Schur bundles of each tensored factor $\mathrm{Sym}^{\ell_1} T_X^* \otimes \cdots \otimes \mathrm{Sym}^{\ell_\kappa} T_X^*$ obtained *e.g.* by an application of Pieri's rule (9) on p. 20 enables one to define a subfiltration to which the same reasoning as on p. 11 applies. □

Thus, as in Rousseau's papers [35, 36] for $n = 3$ and $\kappa = 3$ and as in [29] for $n = 4$ and $\kappa = 4$, the study of the cohomology of the Green-Griffiths bundle $\mathscr{E}_{\kappa,m}^{GG} T_X^*$ is led back to the study of the cohomology of Schur bundles, which might in turn be complicated.

§5. Asymptotic characteristic and asymptotic cohomology

Giambelli determinants of Chern classes. From the lemma on p. 13 and from the theorem on p. 30, we deduce at once from the additivity of Euler-Poincaré characteristic that:

$$\chi\big(X,\ \mathscr{E}_{\kappa,m}^{GG} T_X^*\big) = \sum_{\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n \geqslant 0} M_{\ell_1,\ell_2,\dots,\ell_n}^{\kappa,m} \cdot \chi\big(X,\ \mathscr{S}^{(\ell_1,\ell_2,\dots,\ell_n)} T_X^*\big).$$

But there is a closed asymptotic general formula for:

$$\chi\big(X,\ \mathscr{S}^{(\ell_1,\dots,\ell_n)} T_X^*\big) = (-1)^n\, \chi\big(X,\ \mathscr{S}^{(\ell_1,\dots,\ell_n)} T_X\big),$$

where the $(-1)^n$ comes from $\mathsf{c}_k^* = (-1)^k \mathsf{c}_k$. Recall that a *partition* $(\nu_1, \nu_2, \dots, \nu_n)$ of n is just a collection of nonnegative integers $\nu_1 \geqslant \nu_2 \geqslant \cdots \geqslant \nu_n \geqslant 0$ whose sum $\nu_1 + \nu_2 + \cdots + \nu_n$ equals n.

Theorem. ([29][10]) *The terms of highest order with respect to $|\ell| = \ell_1 + \cdots + \ell_n$ in the Euler-Poincaré characteristic of the Schur bundle $\mathscr{S}^{(\ell_1,\ell_2,\dots,\ell_n)}\, T_X^*$ are homogeneous of order $\frac{n(n+1)}{2}$ and they are given by a sum of determinants indexed by all the partitions $(\nu_1, \dots, \nu_n)$ of n:*

$$(-1)^n\, \chi\Big(X,\ \mathscr{S}^{(\ell_1,\ell_2,\dots,\ell_n)}\, T_X^*\Big) =$$

$$= \sum_{\nu \text{ partition of } n} \frac{\mathsf{C}_{\nu^c}}{(\nu_1 + n - 1)! \cdots \nu_n!} \begin{vmatrix} {\ell_1'}^{\nu_1+n-1} & {\ell_2'}^{\nu_1+n-1} & \cdots & {\ell_n'}^{\nu_1+n-1} \\ {\ell_1'}^{\nu_2+n-2} & {\ell_2'}^{\nu_2+n-2} & \cdots & {\ell_n'}^{\nu_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ {\ell_1'}^{\nu_n} & {\ell_2'}^{\nu_n} & \cdots & {\ell_n'}^{\nu_n} \end{vmatrix} +$$

$$+ \mathsf{O}_n\big(|\ell|^{\frac{n(n+1)}{2}-1}\big),$$

where $\ell_i' := \ell_i + n - i$ for notational brevity, with coefficients C_{ν^c} being expressed in terms of the Chern classes $\mathsf{c}_k = \mathsf{c}_k\big(T_X\big)$ of T_X by means of Giambelli's determinantal expression *depending upon the* conjugate *partition ν^c:*

$$\mathsf{C}_{\nu^c} = \mathsf{C}_{(\nu_1^c,\dots,\nu_n^c)} = \begin{vmatrix} \mathsf{c}_{\nu_1^c} & \mathsf{c}_{\nu_1^c+1} & \mathsf{c}_{\nu_1^c+2} & \cdots & \mathsf{c}_{\nu_1^c+n-1} \\ \mathsf{c}_{\nu_2^c-1} & \mathsf{c}_{\nu_2^c} & \mathsf{c}_{\nu_2^c+1} & \cdots & \mathsf{c}_{\nu_2^c+n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathsf{c}_{\nu_n^c-n+1} & \mathsf{c}_{\nu_n^c-n+2} & \mathsf{c}_{\nu_n^c-n+3} & \cdots & \mathsf{c}_{\nu_n^c} \end{vmatrix},$$

with the understanding, by convention, that $\mathsf{c}_k := 0$ for $k < 0$ or $k > n$, and that $\mathsf{c}_0 := 1$. Furthermore, the remainder $\mathsf{O}_n\big(|\ell|^{\frac{n(n+1)}{2}}\big)$ is a linear combination of homogeneous terms $\mathsf{c}_1^{\tau_1} \mathsf{c}_2^{\tau_2} \cdots \mathsf{c}_n^{\tau_n}$ with $\tau_1 + 2\tau_2 + \cdots + n\tau_n = n$ each multiplied by some polynomial of degree $\leqslant \frac{n(n+1)}{2} - 1$ in the ℓ_i whose coefficients are rational and bounded in absolute value by $\mathsf{Constant}_n$.

---

[10] After [29] was posted on arxiv.org, the author was informed by E. Rousseau that Brückmann's Theorem 4 in [6] entails the above statement and moreover, that it shows how to explicit the remainders.

Because it is elementarily checked that modulo $\mathsf{O}_n\big(|\ell|^{\frac{n(n+1)}{2}-1}\big)$, one has:

$$\begin{vmatrix} {\ell'_1}^{\nu_1+n-1} & {\ell'_2}^{\nu_1+n-1} & \cdots & {\ell'_n}^{\nu_1+n-1} \\ {\ell'_1}^{\nu_2+n-2} & {\ell'_2}^{\nu_2+n-2} & \cdots & {\ell'_n}^{\nu_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ {\ell'_1}^{\nu_n} & {\ell'_2}^{\nu_n} & \cdots & {\ell'_n}^{\nu_n} \end{vmatrix} \equiv \begin{vmatrix} {\ell_1}^{\nu_1+n-1} & {\ell_2}^{\nu_1+n-1} & \cdots & {\ell_n}^{\nu_1+n-1} \\ {\ell_1}^{\nu_2+n-2} & {\ell_2}^{\nu_2+n-2} & \cdots & {\ell_n}^{\nu_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ {\ell_1}^{\nu_n} & {\ell_2}^{\nu_n} & \cdots & {\ell_n}^{\nu_n} \end{vmatrix},$$

we may equivalently replace the ℓ_i'-determinants by the corresponding ℓ_i-determinants in the formula of the theorem. Then for coherence between the above theorem and the computation of the Euler-Poincaré characteristic of $\mathscr{E}_{\kappa,m}^{GG}T_X^*$ conducted independently in Section 3, it should be true that the sum of remainders attached to Schur bundles corresponds to the last remainder of the theorem on p. 13:

$$\sum_{\ell_1\geqslant\ell_2\geqslant\cdots\geqslant\ell_n\geqslant 0} M^{\kappa,m}_{\ell_1,\ell_2,\dots,\ell_n}\cdot\mathsf{O}_n\big(|\ell|^{\frac{n(n+1)}{2}-1}\big) = \mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big).$$

This fact will be established later by the proposition on p. 51 below. Also, using (8) on p. 14, one should consider that all homogeneous products of Chern classes $\mathsf{c}_1^{\tau_1}\cdots\mathsf{c}_n^{\tau_n}$ are implicitly reexpressed in terms of n and d, whence both remainders are in fact of the form $\mathsf{O}_{n,d}\big(|\ell|^{\frac{n(n+1)}{2}-1}\big)$ and $\mathsf{O}_{n,d,\kappa}\big(m^{(\kappa+1)n-2}\big)$.

Dimensions 2, 3 and 4. In greater length, let us for instance write down the expanded sums over partitions, firstly in dimension $n=2$, with two partitions $2=2+0=1+1$:

$$-\chi\big(X,\,\mathscr{S}^{(\ell_1,\ell_2)}T_X^*\big) = \frac{\mathsf{c}_1^2-\mathsf{c}_2}{0!\;3!}\begin{vmatrix}\ell_1^3 & \ell_2^3\\ 1 & 1\end{vmatrix} + \frac{\mathsf{c}_2}{1!\,2!}\begin{vmatrix}\ell_1^2 & \ell_2^2\\ \ell_1 & \ell_2\end{vmatrix} + \mathsf{O}\big(|\ell|^2\big);$$

next in dimension $n=3$, with three partitions $3=3+0+0=2+1+0=1+1+1$:

$$\begin{aligned}\chi\big(X,\,\mathscr{S}^{(\ell_1,\ell_2,\ell_3)}\,T_X^*\big) &= \\ &= \frac{\mathsf{c}_1^3-2\,\mathsf{c}_1\mathsf{c}_2+\mathsf{c}_3}{0!\;1!\;5!}\begin{vmatrix}\ell_1^5 & \ell_2^5 & \ell_3^5\\ \ell_1 & \ell_2 & \ell_3\\ 1 & 1 & 1\end{vmatrix} + \frac{\mathsf{c}_1\mathsf{c}_2-\mathsf{c}_3}{0!\;2!\;4!}\begin{vmatrix}\ell_1^4 & \ell_2^4 & \ell_3^4\\ \ell_1^2 & \ell_2^2 & \ell_3^2\\ 1 & 1 & 1\end{vmatrix} + \\ &\quad + \frac{\mathsf{c}_3}{1!\;2!\;3!}\begin{vmatrix}\ell_1^3 & \ell_2^3 & \ell_3^3\\ \ell_1^2 & \ell_2^2 & \ell_3^2\\ \ell_1 & \ell_2 & \ell_3\end{vmatrix} + \mathsf{O}\big(|\ell|^5\big).\end{aligned}$$

and finally in dimension $n = 4$, with 5 partitions $4 = 4+0+0+0 = 3+1+0+0 = 2+2+0+0 = 2+1+1+0 = 1+1+1+1$:

$$
\begin{aligned}
&\chi\big(X,\, \mathscr{S}^{(\ell_1,\ell_2,\ell_3,\ell_4)}\, T_X^*\big) = \\
&= \frac{\mathsf{c}_1^4 - 3\,\mathsf{c}_1^2\mathsf{c}_2 + \mathsf{c}_2^2 + 2\,\mathsf{c}_1\mathsf{c}_3 - \mathsf{c}_4}{0!\ 1!\ 2!\ 7!} \begin{vmatrix} \ell_1^7 & \ell_2^7 & \ell_3^7 & \ell_4^7 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 & \ell_4^2 \\ \ell_1^1 & \ell_2^1 & \ell_3^1 & \ell_4^1 \\ 1 & 1 & 1 & 1 \end{vmatrix} + \\
&+ \frac{\mathsf{c}_1^2\mathsf{c}_2 - \mathsf{c}_2^2 - \mathsf{c}_1\mathsf{c}_3 + \mathsf{c}_4}{0!\ 1!\ 3!\ 6!} \begin{vmatrix} \ell_1^6 & \ell_2^6 & \ell_3^6 & \ell_4^6 \\ \ell_1^3 & \ell_2^3 & \ell_3^3 & \ell_4^3 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} + \frac{-\mathsf{c}_1\mathsf{c}_3 + \mathsf{c}_2^2}{0!\ 1!\ 4!\ 5!} \begin{vmatrix} \ell_1^5 & \ell_2^5 & \ell_3^5 & \ell_4^5 \\ \ell_1^4 & \ell_2^4 & \ell_3^4 & \ell_4^4 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} + \\
&+ \frac{\mathsf{c}_1\mathsf{c}_3 - \mathsf{c}_4}{0!\ 2!\ 3!\ 5!} \begin{vmatrix} \ell_1^5 & \ell_2^5 & \ell_3^5 & \ell_4^5 \\ \ell_1^3 & \ell_2^3 & \ell_3^3 & \ell_4^3 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 & \ell_4^2 \\ 1 & 1 & 1 & 1 \end{vmatrix} + \frac{\mathsf{c}_4}{1!\ 2!\ 3!\ 4!} \begin{vmatrix} \ell_1^4 & \ell_2^4 & \ell_3^4 & \ell_4^4 \\ \ell_1^3 & \ell_2^3 & \ell_3^3 & \ell_4^3 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 & \ell_4^2 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 \end{vmatrix} + \mathsf{O}\big(|\ell|^9\big).
\end{aligned}
$$

Cohomology of Schur bundles. One could be led to presume that the cohomology dimensions:

$$
h^q = \dim H^q\big(X,\ \mathscr{S}^{(\ell_1,\ell_2,\dots,\ell_n)} T_X^*\big) \qquad\qquad {\scriptstyle (q\,=\,0,\ 1\,\cdots\,n)}
$$

of any Schur bundle over X might be expressed similarly by means of a general formula of the kind:

$$
\begin{aligned}
h^q = \sum_{\tau_1+2\tau_2+\cdots+n\tau_n=n} \mathsf{c}_1^{\tau_1}\mathsf{c}_2^{\tau_2}\cdots\,\mathsf{c}_n^{\tau_n} \sum_{\alpha_1+\alpha_2+\cdots+\alpha_n \leqslant \frac{n(n+1)}{2}} \\
\sum_{\ell_1\geqslant\ell_2\geqslant\cdots\geqslant\ell_n\geqslant 0} h_{\tau_1,\dots,\tau_n;\ \alpha_1,\dots,\alpha_n}^{q;\ \ell_1,\dots,\ell_n} \cdot (\ell_1)^{\alpha_1}(\ell_2)^{\alpha_2}\cdots(\ell_n)^{\alpha_n}
\end{aligned}
$$

involving the Chern classes c_k, the ℓ_i and certain rational coefficients $h_{\tau_1,\dots,\tau_n\,;\ \alpha_1,\dots,\alpha_n}^{q;\ \ell_1,\dots,\ell_n} \in \mathbb{Q}$, or alternatively, after making the substitution (8) on p. 14, as follows:

$$
h^q = \sum_{k=1}^{n+1} d^k \sum_{\alpha_1+\alpha_2+\cdots+\alpha_n \leqslant \frac{n(n+1)}{2}} \ \sum_{\ell_1\geqslant\ell_2\geqslant\cdots\geqslant\ell_n\geqslant 0} h_{k;\ \alpha_1,\dots,\alpha_n}^{q;\ \ell_1,\dots,\ell_n} \cdot (\ell_1)^{\alpha_1}(\ell_2)^{\alpha_2}\cdots(\ell_n)^{\alpha_n}.
$$

However, it turns out to be already known that purely algebraic formulas are certainly impossible, only *semi-algebraic* formulas can be hoped for. Indeed, Brückmann computed in [5] the exact cohomology dimensions:

$$
\dim H^q\big(X,\ \Lambda^r T_X^* \otimes \mathscr{O}_X(t)\big)
$$

for any $q = 0, 1, \dots, n$, any $r = 0, 1, \dots, n$ and any $t \in \mathbb{Z}$, where $\Lambda^r T_X^*$ identifies with $\mathscr{S}^{(1,\dots,1,0,\dots,0)} T_X^*$ (r times 1), and it turns out that the obtained formulas are only piecewise polynomial with respect to the data (n, d, q, r, t). In fact, making

the convention that $\Lambda^0 T_X^* \equiv \mathscr{O}_X(0)$, it is at first well known that:

$$\begin{aligned}
\dim H^0\big(X,\, \mathscr{O}_X(t)\big) &= \tbinom{t+n+1}{n+1} - \tbinom{t+n+1-d}{n+1},\\
\dim H^q\big(X,\, \mathscr{O}_X(t)\big) &= 0 \qquad \text{for all } q \text{ with } 1 \leqslant q \leqslant n-1,\\
\dim H^n\big(X,\, \mathscr{O}_X(t)\big) &= \tbinom{d-n-2-t+n+1}{n+1} - \tbinom{d-n-2-t+n+1-d}{n+1}.
\end{aligned}$$

Using then $\Lambda^n T_X^* = K_X = \mathscr{O}_X(d-n-2)$, one deduces:

$$\begin{aligned}
\dim H^0\big(X,\, \Lambda^n T_X^* \otimes \mathscr{O}_X(t)\big) &= \tbinom{d-n-2+t+n+1}{n+1} - \tbinom{d-n-2+t+n+1-d}{n+1},\\
\dim H^q\big(X,\, \Lambda^n T_X^* \otimes \mathscr{O}_X(t)\big) &= 0 \qquad \text{for all } q \text{ with } 1 \leqslant q \leqslant n-1,\\
\dim H^n\big(X,\, \Lambda^n T_X^* \otimes \mathscr{O}_X(t)\big) &= \tbinom{-t+n+1}{n+1} - \tbinom{-t+n+1-d}{n+1}.
\end{aligned}$$

On the other hand, for $1 \leqslant r \leqslant n-1$, Brückmann ([5]) obtained complete dimension formulas:

$$\begin{aligned}
\dim H^0\big(X,\, \Lambda^r T_X^* \otimes \mathscr{O}_X(t)\big) &= \tbinom{t-1}{r}\tbinom{t+n+1-r}{n+1-r},\\
\dim H^q\big(X,\, \Lambda^r T_X^* \otimes \mathscr{O}_X(t)\big) &= \delta_{q,r}\cdot\delta_{t,0} \qquad \text{for all } q \text{ with } 1 \leqslant q \leqslant n-1 \text{ and } q+r \neq n,\\
\dim H^{n-r}\big(X,\, \Lambda^r T_X^* \otimes \mathscr{O}_X(t)\big) &= \sum_{\mu=0}^{n+2} (-1)^\mu \tbinom{n+2}{\mu}\tbinom{-t-rd-(\mu-1)(d-1)}{n+1} + \delta_{n,2r}\cdot\delta_{t,0},\\
\dim H^n\big(X,\, \Lambda^r T_X^* \otimes \mathscr{O}_X(t)\big) &= \tbinom{-t-1}{n-r}\tbinom{-t+n+1-2r}{n+1-2r}.
\end{aligned}$$

Clearly, these formulas are only semi-algebraic. One does not find in the literature complete formulas for cohomology dimensions of Schur bundles having at least three distinct row lengths.

Majorating the cohomology. Rousseau's strategy developed in [36] and in [17] for dimensions 3 and 4 consists in avoiding exact, probably unfeasible cohomology computations and in substituting for that cohomology *inequalities*.

Let as before X be a geometrically smooth projective algebraic complex hypersurface in $\mathbb{P}^{n+1}(\mathbb{C})$. Let $Fl(T_X^*)$ denote the (complete) flag manifold of T_X^* which organizes as a holomorphic vector bundle $\pi\colon\, Fl(T_X^*) \to X$ of rank $\frac{n(n+1)}{2}$ over X, the fiber of which above an arbitrary point $x \in X$ consists of complete flags:

$$0 = E_{0,x} \subset E_{1,x} \subset \cdots \subset E_{n,x} = T_{X,x},$$

where $\dim E_{i,x} = i$. Let as before $\ell = (\ell_1, \ell_2, \dots, \ell_n)$ with $\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n \geqslant 0$. According to Bott ([3]), there is a canonical *line* bundle $\mathscr{B}^\ell(T_X^*)$ over $Fl(T_X^*)$ with the property that the Schur bundle $\mathscr{S}^{(\ell_1,\dots,\ell_n)}T_X^* \to X$ coincides with the direct image $\pi_*(\mathscr{B}^\ell) \to X$ and whose fiber above an arbitrary flag $E_x \in Fl(T_X^*)$ is $\otimes_{i=1}^n \big(\det(E_{x,i}/E_{x,i-1})\big)^{\otimes \ell_i}$. The fundamental theorem of Bott states that the two bundles $\mathscr{S}^{(\ell_1,\dots,\ell_n)}T_X^*$ and $\mathscr{B}^\ell(T_X^*)$ have the same cohomology, and it is therefore somewhat more convenient to deal with $\mathscr{B}^\ell(T_X^*)$, because *line* bundles are better understood and more studied.

In fact, a certain control of the cohomology by means of inequalities is available thanks to the so-called *Holomorphic Morse inequalities* due to Demailly which state as follows in a general version ([12]) suitable for applications devised by Trapani ([46]). Let $\mathscr{E} \to X$ be a completely arbitrary holomorphic vector bundle of rank $r \geqslant 1$ over a compact Kähler manifold of dimension n, and let $\mathscr{L} \to X$ be a holomorphic line bundle subjected to the specific restriction that it can be written as the difference: $\mathscr{L} = \mathscr{F} \otimes \mathscr{G}^{-1}$ between two line bundles that are ample, or more generally, numerically effective. Then about $\mathscr{L}^k \otimes \mathscr{E}$ as $k \to \infty$, we have the following two collections of asymptotic inequalities, firstly for plain cohomology dimensions, secondly for their alternating sums:

• *Weak Morse inequalities*: For any $q = 0, 1, \dots, n$, one has:

$$h^q\big(X,\ \mathscr{L}^k \otimes \mathscr{E}\big) \leqslant r\, k^n\, \frac{1}{(n-q)!\ q!} \int_X \mathsf{c}_1(\mathscr{F})^{n-q} \cdot \mathsf{c}_1(\mathscr{G})^q + \mathsf{o}(k^n)$$

$$(q = 0,\ 1 \cdots n).$$

• *Strong Morse inequalities*: For any $q = 0, 1, \dots, n$, one has:

$$\sum_{0 \leqslant q' \leqslant q} (-1)^{q-q'}\, h^{q'}\big(X,\ \mathscr{L}^k \otimes \mathscr{E}\big) \leqslant r\, k^n \sum_{0 \leqslant q' \leqslant q} \frac{(-1)^{q-q'}}{(n-q')!\ q'!} \int_X \mathsf{c}_1(\mathscr{F})^{n-q'} \cdot \mathsf{c}_1(\mathscr{G})^{q'} + \\ + \mathsf{o}(k^n).$$

An algebraic proof of these inequalities (without $\mathscr{E}$ and for X projective) by plain induction on dimension but not using any tools from Analysis was given by Angelini in [1]. We then borrow this scheme of proof, as it was applied by Rousseau ([36]) within the Schur bundle context. Weak type inequalities will suffice for us, and the goal is somehow to represent $\mathscr{B}^\ell(T_X^*)$ as a difference between two line bundles that will be positive, hence ample.

To begin with, since $T_X^* \otimes \mathscr{O}_X(2)$ is generated by its global sections, it is semi-positive. According to a general property ([10]), if a holomorphic vector bundle $\mathscr{E} \to X$ is semi-positive, i.e. if $E \geqslant 0$, then the corresponding line bundle $\mathscr{B}^\ell(\mathscr{E})$ is also semi-positive, i.e. $\mathscr{B}^\ell(\mathscr{E}) \geqslant 0$. Applying this to $\mathscr{E} := T_X^* \otimes \mathscr{O}_X(2)$, we get, thanks to a natural isomorphism, that:

$$\mathscr{B}^\ell\big(T_X^* \otimes \mathscr{O}_X(2)\big) \simeq \mathscr{B}^\ell(T_X^*) \otimes \pi^* \mathscr{O}_X(2|\ell|) \geqslant 0 \tag{12}$$

is semi-positive, where $|\ell| = \ell_1 + \cdots + \ell_n$. Tensoring then by $\pi^*\mathscr{O}_X(|\ell|) > 0$, it thus trivially follows that:

$$\mathscr{B}^\ell(T_X^*) \otimes \pi^* \mathscr{O}_X(3|\ell|) > 0$$

is positive. Hence we can write (somehow artificially) $\mathscr{B}^\ell(T_X^*)$, which we will now write $\mathscr{B}^\ell$ for short, as the following difference:

$$\mathscr{B}^\ell = \big[\mathscr{B}^\ell \otimes \pi^* \mathscr{O}_X(3|\ell|)\big] \otimes \big[\pi^* \mathscr{O}_X(3|\ell|)\big]^{-1}$$

between two positive line bundles over $Fl(T_X^*)$, with plainly:

$$\mathscr{F} := \mathscr{B}^\ell \otimes \pi^* \mathscr{O}_X(3|\ell|) \qquad \text{and} \qquad \mathscr{G} := \pi^* \mathscr{O}_X(3|\ell|),$$

in the above notations for Morse inequalities.

Following Angelini and Rousseau, we need even more in order to force the positive cohomologies $H^q\big(Fl(T_X^*),\,\mathscr{F}\big)$, $q=1,\dots,n$, to be vanishing. We remind the Kodaira vanishing theorem which stipulates that, on a projective algebraic complex manifold Z, for every ample line bundle $\mathscr{A}\to X$ one has:

$$0=H^q\big(Z,\,\mathscr{A}\otimes K_Z\big),$$

for all $q=1,\dots,n$. So on the flag manifold $Z:=Fl(T_X^*)$, we not only need that $\mathscr{F}$ be positive (hence ample), but we need also, after decomposing in advance:

$$\mathscr{F}=\big(\mathscr{F}\otimes(K_{Fl(T_X^*)})^{-1}\big)\otimes K_{Fl(T_X^*)},$$

that $\mathscr{A}:=\mathscr{F}\otimes(K_{Fl(T_X^*)})^{-1}$ be positive (hence ample).

For this, we recall at first the known isomorphisms ([3, 10]):

$$\begin{aligned}K_{Fl(T_X^*)}&\simeq\big[\mathscr{B}^{2n-1,\dots,3,1}\big]^{-1}\otimes\pi^*(K_X)^{\otimes(n+1)}\\&\simeq\big[\mathscr{B}^{2n-1,\dots,3,1}\big]^{-1}\otimes\pi^*\mathscr{O}_X\big((n+1)(d-n-2)\big),\end{aligned}$$

from which we hence deduce:

$$\begin{aligned}\mathscr{F}\otimes(K_{Fl(T_X^*)})^{-1}&\simeq\mathscr{B}^{\ell}\otimes\mathscr{B}^{2n-1,\dots,3,1}\otimes\pi^*\mathscr{O}_X\big(3|\ell|-(n+1)(d-n-2)\big)\\&\simeq\mathscr{B}^{\ell_1+2n-1,\dots,\ell_{n-1}+3,\ell_n+1}\otimes\pi^*\mathscr{O}_X\big(3|\ell|-(n+1)(d-n-2)\big).\end{aligned}$$

But similarly as in (12) a short while ago, we know that the bundle:

$$\mathscr{B}^{\ell_1+2n-1,\dots,\ell_{n-1}+3,\ell_n+1}\otimes\pi^*\mathscr{O}_X\big(2[\ell_1+2n-1+\cdots+\ell_{n-1}+3+\ell_n+1]\big)$$

is semi-positive, whence it is surely positive after it is tensored only by $\pi^*\mathscr{O}_X(1)$. Consequently, observing $2n-1+\cdots+3+1=n^2$, our bundle $\mathscr{F}\otimes(K_{Fl(T_X^*)})^{-1}$ will be positive when:

$$3|\ell|-(n+1)(d-n-2)\geqslant 1+2(|\ell|+n^2),$$

that is to say when:

$$|\ell|\geqslant 1+2n^2+(n+1)(d-n-2),$$

or with less effective information, when $|\ell|\geqslant\mathsf{Constant}_{n,d}$. Under this restriction concerning $|\ell|$ which insures the applicability of Kodaira's vanishing theorem, Rousseau's scheme of proof works in arbitrary dimension n (cf. [39] and also [17] for the case $n=4$), and it yields the following majorations:

$$\begin{aligned}h^q\big(X,\,\mathscr{S}^{(\ell_1,\dots,\ell_n)}T_X^*\big)\leqslant\chi\big(X,\,&\mathscr{S}^{(\ell_1,\dots,\ell_n)}T_X^*\otimes\mathscr{O}_X(3(q+1)|\ell|)\big)-\\&-\tbinom{q}{1}\,\chi\big(X,\,\mathscr{S}^{(\ell_1,\dots,\ell_n)}T_X^*\otimes\mathscr{O}_X(3q|\ell|)\big)+\\&+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots+\\&+(-1)^{q-1}\tbinom{q}{q-1}\,\chi\big(X,\,\mathscr{S}^{(\ell_1,\dots,\ell_n)}T_X^*\otimes\mathscr{O}_X(6|\ell|)\big)+\\&+(-1)^q\tbinom{q}{q}\,\chi\big(X,\,\mathscr{S}^{(\ell_1,\dots,\ell_n)}T_X^*\otimes\mathscr{O}_X(3|\ell|)\big),\end{aligned}$$

in terms of alternating sums of Euler-Poincaré characteristics. Applying Brückmann's formula for the explicit computation of the appearing Euler-Poincaré characteristics (Theorem 4 in [6]), we then get the following result.

Theorem. *Let $X = X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a geometrically smooth projective algebraic complex hypersurface of general type, i.e. of degree $d \geqslant n+3$, and let $\ell = (\ell_1, \ldots, \ell_{n-1}, \ell_n)$ with $\ell_1 \geqslant \cdots \geqslant \ell_{n-1} \geqslant \ell_n \geqslant 0$. If:*

$$|\ell| = \ell_1 + \cdots + \ell_{n-1} + \ell_n \geqslant \mathsf{Constant}_{n,d},$$

then for every $q = 1, 2, \ldots, n$, the dimensions of the positive cohomology groups of the Schur bundle $\mathscr{S}^{(\ell_1,\ldots,\ell_{n-1},\ell_n)}T_X^*$ over X satisfy a general majoration of the form:

$$h^q\big(X,\ \mathscr{S}^{(\ell_1,\ldots,\ell_{n-1},\ell_n)}T_X^*\big) \leqslant \mathsf{Constant}_{n,d} \prod_{1\leqslant i<j\leqslant n} (\ell_i - \ell_j) \Bigg[\sum_{\beta_1+\cdots+\beta_{n-1}+\beta_n=n} \ell_1^{\beta_1} \cdots \ell_{n-1}^{\beta_{n-1}} \ell_n^{\beta_n} \Bigg] + \\ + \mathsf{Constant}_{n,d} \Bigg[\sum_{\alpha_1+\cdots+\alpha_n \leqslant \frac{n(n+1)}{2}-1} \ell_1^{\alpha_1} \cdots \ell_n^{\alpha_n} \Bigg],$$

with leading terms being homogeneous of degree $\frac{n(n+1)}{2}$ with respect to the ℓ_i and divisible by all the differences $(\ell_i - \ell_j)$, where $1 \leqslant i < j \leqslant n$.

For the estimates that we will conduct in the next sections, we need none of the three $\mathsf{Constant}_{n,d}$ above to be effective. Admitting this, raising if necessary the two $\mathsf{Constants}_{n,d}$ appearing in the right-hand side, it follows that the majoration is in fact valid for every ℓ, since the restriction that $|\ell|$ be large enough can obviously be absorbed by the $\mathsf{Constants}_{n,d}$. Also, one must observe that the Euler-Poincaré characteristic provided by the theorem on p. 31 satisfies the same kind of majoration, hence *all* cohomology dimensions $h^0, h^1, \ldots, h^n$ do the same. However, we want to underline that, even with an effective control on $\mathsf{Constant}_{n,d}$ similar as in [36, 17] for $n = 3$ and $n = 4$, the above kind of majoration cannot at all conduct to the optimal degree bound $d \geqslant n+3$ of the Main Theorem, because we will see that the presence of the monomial ℓ_n^n in $\sum_\beta \ell^\beta$ forces to lose a nonzero portion of the $(\log \kappa)^n$ entering in the Euler-Poincaré characteristic when summing $\sum M_\ell^{\kappa,m} \mathscr{S}^{(\ell)}$ over Schur bundles.

§6. Emergence of basic numerical sums

Expanding and rewriting. At least, the explicit formula for the Euler characteristic and the cohomology bounds for Schur bundles firmly motivates to consider basic numerical sums of the form:

$$\sum_{\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n \geqslant 0} M_{\ell_1,\ell_2,\ldots,\ell_n}^{\kappa,m} \, \ell_1^{\alpha_1} \ell_2^{\alpha_2} \cdots \ell_n^{\alpha_n},$$

for any n-tuple of nonnegative integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ of length $\leqslant \frac{n(n+1)}{2} - 1$ if remainders are to be taken into consideration, or else of length $|\alpha| = \alpha_1 + \cdots + \alpha_n$ constant equal to $\frac{n(n+1)}{2}$, if major terms are to be studied. After some reflections based on manuscript explorations and on intense thought, it appears

a posteriori convenient, if not adequate, to express all quantities in terms of the successive differences between horizontal lengths in the Young diagram:

$$\ell_1 - \ell_2, \qquad \ell_2 - \ell_3, \qquad \cdots\cdots, \qquad \ell_{n-1} - \ell_n, \qquad \ell_n,$$

that is to say, in terms of the horizontal lengths of the appearing successive blocks of constant depths. Thus accordingly, we may rewrite any appearing monomial $\ell_1^{\alpha_1} \cdots \ell_n^{\alpha_n}$ by inserting differences as follows:

$$\begin{aligned}\ell_1^{\alpha_1}\ell_2^{\alpha_2}\cdots\ell_{n-1}^{\alpha_{n-1}}\ell_n^{\alpha_n} = \big(\ell_1 - \ell_2 + \ell_2 - \ell_3 + \cdots + \ell_{n-1} - \ell_n + \ell_n\big)^{\alpha_1}\cdot \\ \cdot\big(\ell_2 - \ell_3 + \cdots + \ell_{n-1} - \ell_n + \ell_n\big)^{\alpha_2}\cdot \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdot\big(\ell_{n-1} - \ell_n + \ell_n\big)^{\alpha_{n-1}}\cdot \\ \cdot\big(\ell_n\big)^{\alpha_n},\end{aligned}$$

and then we may simply expand all the appearing powers to obtain a certain sum, with integer integer coefficients, of interesting monomials of the specific form:

$$\big(\ell_1 - \ell_2\big)^{\alpha_1'}\big(\ell_2 - \ell_3\big)^{\alpha_2'}\cdots\cdots\big(\ell_{n-1} - \ell_n\big)^{\alpha_{n-1}'}\big(\ell_n\big)^{\alpha_n'},$$

the total degree in the ℓ_i being evidently preserved:

$$\alpha_1' + \alpha_2' + \cdots + \alpha_{n-1}' + \alpha_n' = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

Only multinomial coefficients being involved in the expansion, we have a simple inequality of the form:

(13)

$$\ell_1^{\alpha_1}\cdots\ell_{n-1}^{\alpha_{n-1}}\ell_n^{\alpha_n} \leqslant \mathsf{Constant}_n\cdot \sum_{\alpha_1'+\cdots+\alpha_{n-1}'+\alpha_n'=\alpha_1+\cdots+\alpha_{n-1}+\alpha_n} \big(\ell_1-\ell_2\big)^{\alpha_1'}\cdots\big(\ell_{n-1}-\ell_n\big)^{\alpha_{n-1}'}\big(\ell_n\big)^{\alpha_n'}.$$

Basic numerical sums. As a consequence, in order to verify that the contribution in $\sum M_\ell^{\kappa,m}\cdot\ell^\alpha$ of any monomial $\ell_1^{\alpha_1}\cdots\ell_{n-1}^{\alpha_{n-1}}\ell_n^{\alpha_n}$ of total degree:

$$\alpha_1 + \cdots + \alpha_{n-1} + \alpha_n \leqslant \tfrac{n(n+1)}{2} - 1$$

which possibly appears in a general remainder of the form $\mathsf{O}_{n,d}\big(|\ell|^{\frac{n(n+1)}{2}-1}\big)$ still falls into the corresponding m-remainder $\mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big)$, we are led back to studying the asymptotic behavior, as $m\to\infty$, of basic numerical sums of the general form:

$$\sum_{\ell_1\geqslant\ell_2\geqslant\cdots\geqslant\ell_{n-1}\geqslant\ell_n\geqslant 0} M_{\ell_1,\ell_2,\dots,\ell_{n-1},\ell_n}^{\kappa,m}\cdot\big(\ell_1-\ell_2\big)^{\alpha_1'}\cdots\cdots\big(\ell_{n-1}-\ell_n\big)^{\alpha_{n-1}'}\big(\ell_n\big)^{\alpha_n'},$$

with again $\alpha_1' + \cdots + \alpha_{n-1}' + \alpha_n' \leqslant \frac{n(n+1)}{2} - 1$, and now, everything has become purely combinatorial, that is to say, complex geometry concepts have entirely disappeared.

On the other hand, after expanding any $\prod_{i<j}(\ell_i - \ell_j)\,\ell^\beta$ with $|\beta| = n$ appearing both as principal terms in the Euler-Poincaré characteristic of $\mathscr{S}^{(\ell)}T_X^*$

and in the cohomology majorations provided by the theorem on p. 37, and after performing the rewriting in terms of $\ell_1 - \ell_2, \ldots, \ell_{n-1} - \ell_n, \ell_n$ as above, we are led back to estimating the same kind of basic numerical sums, but this time with $\alpha'_1 + \cdots + \alpha'_{n-1} + \alpha'_n = \frac{n(n+1)}{2}$.

Since we will not attempt to compute, even asymptotically, the multiplicities $M^{\kappa,m}_{\ell_1,\ldots,\ell_n}$ for which only semi-algebraic formulas could exist as an examination for small values of n and κ shows, we will rewrite such basic numerical sums under the following more archetypal form[11]:

(14)

$$\boxed{\begin{aligned}&\sum_{\ell_1\geqslant\ell_2\geqslant\cdots\geqslant\ell_{n-1}\geqslant\ell_n\geqslant 0} M^{\kappa,m}_{\ell_1,\ell_2,\ldots,\ell_{n-1},\ell_n}\cdot\big(\ell_1-\ell_2\big)^{\alpha'_1}\cdots\cdots\big(\ell_{n-1}-\ell_n\big)^{\alpha'_{n-1}}\big(\ell_n\big)^{\alpha'_n} = \\ &= \sum_{\substack{\mathsf{YT}\text{ semi}-\text{standard}\\ \mathsf{weight}(\mathsf{YT})=m}} \big(\ell_1(\mathsf{YT})-\ell_2(\mathsf{YT})\big)^{\alpha'_1}\cdots\big(\ell_{n-1}(\mathsf{YT})-\ell_n(\mathsf{YT})\big)^{\alpha'_{n-1}}\big(\ell_n(\mathsf{YT})\big)^{\alpha'_n}\end{aligned}},$$

where in the second line, YT runs over all the possible semi-standard Young tableaux, where $\ell_i(\mathsf{YT})$ denote the length of the i-th line of YT, and where as before $\mathsf{weight}(\mathsf{YT})$ denotes the total number of primes appearing in the associated Δ-monomial, that is to say, the sum of all the λ_i^j occupying the squares of YT:

$$\begin{aligned}\mathsf{weight}\big(\mathsf{YT}\big) &= \mathsf{weight}\big(\mathsf{YD}_{(\ell_1,\ldots,\ell_n)}(\lambda_i^j)\big)\\ &= \sum_{1\leqslant j_1\leqslant\ell_1}\lambda_1^{j_1} + \sum_{1\leqslant j_2\leqslant\ell_2}\lambda_2^{j_2} + \cdots + \sum_{1\leqslant j_n\leqslant\ell_n}\lambda_n^{j_n}.\end{aligned}$$

Then more tractable computations and partially explicit formulas will come up as being somewhat available in the next sections.

Thus, assuming from now on that $\kappa \geqslant n$ is at least equal to the dimension, our first main goal will be to establish (corollary on p. 53 below) that for every $(\alpha'_1,\ldots,\alpha'_{n-1},\alpha'_n)\in\mathbb{N}^n$, the following precise logarithmic-like majoration holds:

$$\begin{aligned}&\sum_{\substack{\mathsf{YT}\text{ semi}-\text{standard}\\ \mathsf{weight}(\mathsf{YT})=m}} \big(\ell_1(\mathsf{YT})-\ell_2(\mathsf{YT})\big)^{\alpha'_1}\cdots\big(\ell_{n-1}(\mathsf{YT})-\ell_n(\mathsf{YT})\big)^{\alpha'_{n-1}}\big(\ell_n(\mathsf{YT})\big)^{\alpha'_n}\leqslant\\ &\qquad\leqslant \mathsf{Constant}_{n,\kappa}\cdot m^{\alpha'_1+\cdots+\alpha'_{n-1}+\alpha'_n}\cdot m^{n\kappa-\frac{n(n-1)}{2}}.\end{aligned}$$

Applying these majorations when $|\alpha'|\leqslant\frac{n(n+1)}{2}-1$, it will then follow in particular that the right-hand side majorant is an $\mathsf{O}_{n,\kappa}\big(m^{(n+1)\kappa-2}\big)$, whence remainders match through summation in Euler-Poincaré characteristics, as was announced a bit after the theorem on p. 31.

[11] The equality written follows immediately from the definitions: the passage from the second line to the first line just consists in counting the semi-standard Young Tableaux of weight m which have the same underlying Young diagram $\mathsf{YD}_{(\ell_1,\ell_2,\ldots,\ell_{n-1},\ell_n)}$, and their number is just what we denoted by the multiplicity $M^{\kappa,m}_{\ell_1,\ell_2,\ldots,\ell_{n-1},\ell_n}$.

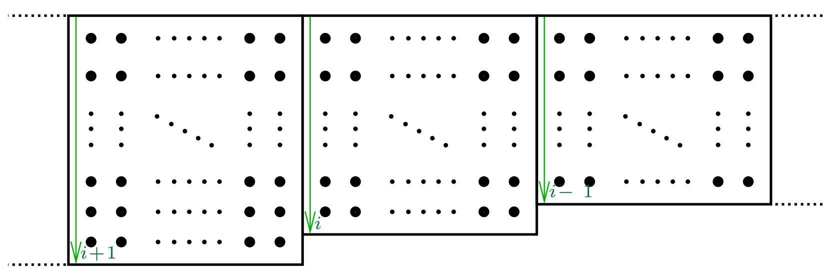
Afterward, we will study what arises when $\alpha_1' + \cdots + \alpha_{n-1}' + \alpha_n' = \frac{n(n+1)}{2}$. In any case, we need to analyze more deeply what comes out from the semi-standard Young tableaux of weight m.

§7. Asymptotic combinatorics of semi-standard Young tableaux

Repetitions in the Δ-monomials. In the general Δ-monomial modulo the Plücker relations given by (11) on p. 29:

$$\prod_{d_1 \geqslant i \geqslant 1} \prod_{1+\ell_{i+1} \leqslant j \leqslant \ell_i} \Delta_{1,\dots,i}^{\lambda_1^j,\dots,\lambda_i^j},$$

there may exist (several) repetitions of a given determinant $\Delta_{1,\dots,i}^{\lambda_1^j,\dots,\lambda_i^j}$, since in the semi-standard Young tableau, the increasing property enjoyed by the λ_i^j is only weak along its rows. So in $\mathsf{YD}_{(\ell_1,\dots,\ell_n)}(\lambda_i^j)$, we should describe with more explicit information the typical block of depth i:



which is naturally located between a block of depth $i+1$ on its left, and a block of depth $i-1$ on its right. To this aim, let us rewrite such a block as follows:

$$\boxed{\begin{array}{ccc} \lambda_1^{1+\ell_{i+1}} & \cdots & \lambda_1^{\ell_i} \\ \lambda_2^{1+\ell_{i+1}} & \cdots & \lambda_2^{\ell_i} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \lambda_i^{1+\ell_{i+1}} & \cdots & \lambda_i^{\ell_i} \end{array}} = \boxed{\begin{bmatrix} \mu_1^j \\ \mu_2^j \\ \cdot\cdot \\ \mu_i^j \end{bmatrix}^{a_{\mu_1^j,\mu_2^j,\dots,\mu_i^j}} \cdots \begin{bmatrix} \lambda_1^j \\ \lambda_2^j \\ \cdot\cdot \\ \lambda_i^j \end{bmatrix}^{a_{\lambda_1^j,\lambda_2^j,\dots,\lambda_i^j}} \cdots \begin{bmatrix} \nu_1^j \\ \nu_2^j \\ \cdot\cdot \\ \nu_i^j \end{bmatrix}^{a_{\nu_1^j,\nu_2^j,\dots,\nu_i^j}}}.$$

Here firstly, looking at the two extreme (right and left) columns, we changed the notation for later purposes, denoting $\mu_l^j := \lambda_l^{1+\ell_{i+1}}$ and $\nu_l^j := \lambda_l^j$ for any row index $l = 1, 2, \dots, i$; secondly, the appearing exponents $a_{*,*,\dots,*}$ are meant to denote repetitions of (bracketed) columns, so that naturally their sum equals the horizontal length of the initially considered i-th block:

$$\ell_i - \ell_{i+1} = a_{\mu_1^j,\mu_2^j,\dots,\mu_i^j} + \cdots + a_{\lambda_1^j,\lambda_2^j,\dots,\lambda_i^j} + \cdots + a_{\nu_1^j,\nu_2^j,\dots,\nu_i^j};$$

thirdly and lastly, the succession of columns now increases strictly when one disregards the repetitions:

$$\begin{bmatrix}\mu_1^j\\ \mu_2^j\\ \cdot\cdot\\ \mu_i^j\end{bmatrix} < \cdots < \begin{bmatrix}\lambda_1^j\\ \lambda_2^j\\ \cdot\cdot\\ \lambda_i^j\end{bmatrix} < \cdots < \begin{bmatrix}\nu_1^j\\ \nu_2^j\\ \cdot\cdot\\ \nu_i^j\end{bmatrix},$$

where by definition we declare that a column $(\lambda_l')_{1\leqslant l\leqslant i}$ is smaller (strictly) than another column $(\lambda_l'')_{1\leqslant l\leqslant i}$ if all its row elements are smaller (weakly): $\lambda_l'\leqslant\lambda_l''$ for $l=1,\dots,i$, *and if* there exists at least one row index l_0 for which $\lambda_{l_0}'<\lambda_{l_0}''$. As a result, we have represented our typical semi-standard Young tableau of depth d_1 as follows by emphasizing precisely the column repetitions, all the appearing columns being now pairwise distinct and ordered increasingly:

$$\boxed{\begin{bmatrix}\mu_1^{d_1}\\ \mu_2^{d_1}\\ \mu_3^{d_1}\\ \cdot\\ \cdot\\ \mu_{d_1-1}^{d_1}\\ \mu_{d_1}^{d_1}\end{bmatrix}^* \cdots \begin{bmatrix}\nu_1^{d_1}\\ \nu_2^{d_1}\\ \nu_3^{d_1}\\ \cdot\\ \cdot\\ \nu_{d_1-1}^{d_1}\\ \nu_{d_1}^{d_1}\end{bmatrix}^*}\boxed{\begin{bmatrix}\mu_1^{d_1-1}\\ \mu_2^{d_1-1}\\ \mu_3^{d_1-1}\\ \cdot\\ \cdot\\ \mu_{d_1-1}^{d_1-1}\end{bmatrix}^* \cdots \begin{bmatrix}\nu_1^{d_1-1}\\ \nu_2^{d_1-1}\\ \nu_3^{d_1-1}\\ \cdot\\ \cdot\\ \nu_{d_1-1}^{d_1-1}\end{bmatrix}^*}\cdots\boxed{\begin{bmatrix}\mu_1^3\\ \mu_2^3\\ \mu_3^3\end{bmatrix}^* \cdots \begin{bmatrix}\nu_1^3\\ \nu_2^3\\ \nu_3^3\end{bmatrix}^*}\boxed{\begin{bmatrix}\mu_1^2\\ \mu_2^2\end{bmatrix}^* \cdots \begin{bmatrix}\nu_1^2\\ \nu_2^2\end{bmatrix}^*}\boxed{\begin{bmatrix}\mu_1^1\end{bmatrix}^* \cdots \begin{bmatrix}\nu_1^1\end{bmatrix}^*}.$$

Here, for reasons of space, the multiplicities $*$ are not written in length, but as above, they should be read for a typical column as an integer $a_{\lambda_1^j,\lambda_2^j,\dots,\lambda_i^j}$ depending on the column which is $\geqslant 1$; so we understand that the multiplicities of appearing columns are always positive, but it may well happen that some blocks of given depths are completely missing[12], so that at some places, there are contacts between a block of depth, say $i+c$ on the left for some $c\geqslant 2$, and a block of depth i on the right. Furthermore, inside any block, semi-standard inequalities must hold, and between the two contacting columns of two neighboring blocks, say of depth $i+1$ and of depth i:

$$\cdots\boxed{\begin{bmatrix}\mu_1^{i+1}\\ \mu_2^{i+1}\\ \cdot\\ \cdot\\ \mu_i^{i+1}\\ \mu_{i+1}^{i+1}\end{bmatrix}^* \cdots \begin{bmatrix}\nu_1^{i+1}\\ \nu_2^{i+1}\\ \cdot\\ \cdot\\ \nu_i^{i+1}\\ \nu_{i+1}^{i+1}\end{bmatrix}^*}\boxed{\begin{bmatrix}\mu_1^{i}\\ \mu_2^{i}\\ \cdot\\ \cdot\\ \mu_i^{i}\end{bmatrix}^* \cdots \begin{bmatrix}\nu_1^{i}\\ \nu_2^{i}\\ \cdot\\ \cdot\\ \nu_i^{i}\end{bmatrix}^*}\cdots,$$

[12] However, in what we will be interested in later for applications to the Green-Griffiths bundle of jets, we will have to study only Young diagrams $\mathsf{YD}_{(\ell_1,\dots,\ell_n)}$ for which $\ell_1-\ell_2,\dots,\ell_{n-1}-\ell_n$ and ℓ_n are all positive, and even in fact large, so letting all blocks appear in diagrams is harmless.

there must of course in addition exist the semi-standard-like truncated inequalities:

$$\begin{array}{ccc} \nu_1^{i+1} & \leqslant & \mu_1^i \\ \nu_2^{i+1} & \leqslant & \mu_2^i \\ \cdot & \leqslant & \cdot \\ \nu_i^{i+1} & \leqslant & \mu_i^i \\ \nu_{i+1}^{i+1}, & & \end{array} \tag{15}$$

with nothing about the last element of the longest column; if more generally, the contact holds between a nonvoid block of depth $i+c$ on the left and a nonvoid block of depth i on the right, in the case where blocks of the intermediate depths $i+c-1, \dots, i+1$ are inextant, then the last c elements $\nu_{i+1}^{i+c}, \dots, \nu_{i+c}^{i+c}$ of the rightmost column of the longest block located on the left are subjected to no constraint at all.

A notable example of such a semi-standard Young tableau representing a Δ-monomial written in such a way more informative than before is the following:

$$\underbrace{\boxed{\begin{bmatrix}1\\2\\3\\\cdot\\\cdot\\n-1\\n\end{bmatrix}^* \cdots \begin{bmatrix}1\\2\\3\\\cdot\\\cdot\\n-1\\\kappa\end{bmatrix}^*}}_{\kappa-n+1} \underbrace{\boxed{\begin{bmatrix}1\\2\\3\\\cdot\\\cdot\\n-1\end{bmatrix}^* \cdots \begin{bmatrix}1\\2\\3\\\cdot\\\cdot\\\kappa\end{bmatrix}^*}}_{\kappa-n+2} \cdots \underbrace{\boxed{\begin{bmatrix}1\\2\\3\end{bmatrix}^* \cdots \begin{bmatrix}1\\2\\\kappa\end{bmatrix}^*}}_{\kappa-2} \underbrace{\boxed{\begin{bmatrix}1\\2\end{bmatrix}^* \cdots \begin{bmatrix}1\\\kappa\end{bmatrix}^*}}_{\kappa-1} \underbrace{\boxed{\begin{bmatrix}1\end{bmatrix}^* \cdots \begin{bmatrix}\kappa\end{bmatrix}^*}}_{\kappa} .$$

It has depth $d_1 = n$ and its first column on the left corresponds naturally to the n-dimensional Wronskian:

$$\Delta_{1,2,3,\dots,n-1,n}^{1,2,3,\dots,n-1,n} = \begin{vmatrix} f_1' & f_2' & f_3' & \cdots & f_{n-1}' & f_n' \\ f_1'' & f_2'' & f_3'' & \cdots & f_{n-1}'' & f_n'' \\ f_1''' & f_2''' & f_3''' & \cdots & f_{n-1}''' & f_n''' \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ f_1^{(n-1)} & f_2^{(n-1)} & f_3^{(n-1)} & \cdots & f_{n-1}^{(n-1)} & f_n^{(n-1)} \\ f_1^{(n)} & f_2^{(n)} & f_3^{(n)} & \cdots & f_{n-1}^{(n)} & f_n^{(n)} \end{vmatrix}$$

raised to a certain power $* = a_{1,2,3,\dots,n-1,n} \geqslant 1$; in its first block, the bottom indices of extant columns are $n, n+1, \dots, \kappa-1, \kappa$ while all other indices above are constant horizontally; in its second block, the bottom indices of extant columns are $n-1, n, n+1, \dots, \kappa+1, \kappa$; and so on. Therefore, the number of pairwise

distinct columns is equal to[13]:

$$(\kappa - n + 1) + (\kappa - n + 2) + \cdots + (\kappa - 2) + (\kappa - 1) + \kappa = n\kappa - \tfrac{n(n-1)}{2}.$$

Observe that this example of semi-standard Young tableau constitutes not just one Δ-monomial, but in fact it represents a whole specific *family* of Δ-monomials which are labelled by all possible column exponents $* \geqslant 1$. Of course, the row lengths, or equivalently and what is clearer, the differences of row lengths, may, as already seen, be expressed in terms of these exponents:

$$\left\{\begin{aligned} \ell_n &= a_{1,2,3,\dots,n-1,n} + \cdots + a_{1,2,3,\dots,n-1,\kappa} \\ \ell_{n-1} - \ell_n &= a_{1,2,3,\dots,n-1} + \cdots + a_{1,2,3,\dots,\kappa} \\ \cdot\cdot \quad \cdot\cdot &= \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \ell_3 - \ell_4 &= a_{1,2,3} + \cdots + a_{1,2,\kappa} \\ \ell_2 - \ell_3 &= a_{1,2} + \cdots + a_{1,\kappa} \\ \ell_1 - \ell_2 &= a_1 + \cdots + a_\kappa. \end{aligned}\right.$$

One should also remember from the theorem on p. 30 that the total number of primes in any considered Δ-monomial should be constant equal to m, a condition that can now be read here as:

$$\begin{aligned} m = {} & \big[1+2+3+\cdots+n-1+n\big]a_{1,2,3,\dots,n-1,n} + \cdots + \big[1+2+3+\cdots+n-1+\kappa\big]a_{1,2,3,\dots,n-1,\kappa} + \\ & + \big[1+2+3+\cdots+n-1\big]a_{1,2,3,\dots,n-1} + \cdots + \big[1+2+3+\cdots+\kappa\big]a_{1,2,3,\dots,\kappa} + \\ & + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + \\ & + \big[1+2+3\big]a_{1,2,3} + \cdots + \big[1+2+\kappa\big]a_{1,2,\kappa} + \\ & + \big[1+2\big]a_{1,2} + \cdots + \big[1+\kappa\big]a_{1,\kappa} + \\ & + \big[1\big]a_1 + \cdots + \big[\kappa\big]a_\kappa; \end{aligned}$$

in this equation, each exponent $a_{1,2,3,\dots,i-1,\lambda}$, $i \leqslant \lambda \leqslant \kappa$, is multiplied by the sum $1 + 2 + 3 + \cdots + i - 1 + \lambda$ of its lower indices, an integer which equals the total number of primes in the determinant $\Delta^{1,2,3,\dots,i-1,\lambda}_{1,2,3,\dots,i-1,i}$, so that the total number of primes in the power $\big[\Delta^{1,2,3,\dots,i-1,\lambda}_{1,2,3,\dots,i-1,i}\big]^{a_{1,2,3,\dots,i-1,\lambda}}$ is indeed equal to the product:

$$\big[1 + 2 + 3 + \cdots + i - 1 + \lambda\big]a_{1,2,3,\dots,i-1,\lambda}.$$

Maximal number of pairwise distinct columns in $\mathsf{YD}_{(\ell_1,\dots,\ell_n)}(\lambda_i^j)$. We now come back to a general semi-standard Young tableau of depth d_1 with extreme

[13] Notice *passim* that this number minus 1 plus the (constant) degree of any homogeneous monomial $\big(\ell_1 - \ell_2\big)^{\alpha_1} + \cdots + \big(\ell_{n-1} - \ell_n\big)^{\alpha_{n-1}}\big(\ell_n\big)^{\alpha_n}$:

$$n\kappa - \tfrac{n(n-1)}{2} - 1 + \tfrac{n(n+1)}{2} = (\kappa + 1)n - 1$$

equals the exponent of m in the formula of the theorem on p. 13 about the asymptotic behavior of the Euler-Poincaré characteristic.

columns in each block that can be arbitrary:

(16)

$$\underbrace{\boxed{\begin{bmatrix}\mu_1^{d_1}\\ \mu_2^{d_1}\\ \mu_3^{d_1}\\ \cdot\\ \cdot\\ \mu_{d_1-1}^{d_1}\\ \mu_{d_1}^{d_1}\end{bmatrix}^* \cdots \begin{bmatrix}\nu_1^{d_1}\\ \nu_2^{d_1}\\ \nu_3^{d_1}\\ \cdot\\ \cdot\\ \nu_{d_1-1}^{d_1}\\ \nu_{d_1}^{d_1}\end{bmatrix}^*}}_{\#\leqslant D_{d_1}} \underbrace{\boxed{\begin{bmatrix}\mu_1^{d_1-1}\\ \mu_2^{d_1-1}\\ \mu_3^{d_1-1}\\ \cdot\\ \cdot\\ \mu_{d_1-1}^{d_1-1}\end{bmatrix}^* \cdots \begin{bmatrix}\nu_1^{d_1-1}\\ \nu_2^{d_1-1}\\ \nu_3^{d_1-1}\\ \cdot\\ \cdot\\ \nu_{d_1-1}^{d_1-1}\end{bmatrix}^*}}_{\#\leqslant D_{d_1-1}} \cdots \underbrace{\boxed{\begin{bmatrix}\mu_1^{3}\\ \mu_2^{3}\\ \mu_3^{3}\end{bmatrix}^* \cdots \begin{bmatrix}\nu_1^{3}\\ \nu_2^{3}\\ \nu_3^{3}\end{bmatrix}^*}}_{\#\leqslant D_3} \underbrace{\boxed{\begin{bmatrix}\mu_1^{2}\\ \mu_2^{2}\end{bmatrix}^* \cdots \begin{bmatrix}\nu_1^{2}\\ \nu_2^{2}\end{bmatrix}^*}}_{\#\leqslant D_2} \underbrace{\boxed{\begin{bmatrix}\mu_1^{1}\end{bmatrix}^* \cdots \begin{bmatrix}\nu_1^{1}\end{bmatrix}^*}}_{\#\leqslant D_1}.$$

What is the maximal possible number of pairwise distinct $*$-ed columns? In the rightmost block, the number of entries in the single row is clearly less than or equal to:

$$D_1 := 1 + \nu_1^1 - \mu_1^1.$$

In full generality, there may well be several gaps from μ_1^1 to ν_1^1 in the 'cdots', so D_1 is an upper bound. If a general $*$-ed column of depth i with $1 \leqslant i \leqslant d_1$ has two immediate neighbors, with no possible intermediate neighbors:

$$\begin{bmatrix}\lambda_1'\\ \lambda_2'\\ \cdot\cdot\\ \lambda_i'\end{bmatrix}^* < \begin{bmatrix}\lambda_1\\ \lambda_2\\ \cdot\cdot\\ \lambda_i\end{bmatrix}^* < \begin{bmatrix}\lambda_1''\\ \lambda_2''\\ \cdot\cdot\\ \lambda_i''\end{bmatrix}^*,$$

then necessarily the two sums of horizontal differences:

$$\begin{aligned}\lambda_1 - \lambda_1' + \lambda_2 - \lambda_2' + \cdots + \lambda_i - \lambda_i' &= 1\\ \lambda_1'' - \lambda_1 + \lambda_2'' - \lambda_2 + \cdots + \lambda_i'' - \lambda_i &= 1\end{aligned}$$

are both smallest possible, equal to 1. It follows easily from this observation and from the constraint of semi-standarcy that in the penultimate block, in the antepenultimate block and in the general i-th block, the numbers of pairwise distinct $*$-ed columns are at most equal to, respectively:

$$\begin{aligned}D_2 &:= 1 + (\nu_1^2 - \mu_1^2) + (\nu_2^2 - \mu_2^2),\\ D_3 &:= 1 + (\nu_1^3 - \mu_1^3) + (\nu_2^3 - \mu_2^3) + (\nu_3^3 - \mu_3^3),\end{aligned}$$

and to:

$$D_i := 1 + \sum_{l=1}^{i} (\nu_l^i - \mu_l^i). \tag{17}$$

Consequently, the total number of pairwise distinct columns in an arbitrary semi-standard Young tableau is at most equal to $D_1 + D_2 + D_3 + \cdots + D_{d_1-1} + D_{d_1}$,

that is to say to:

$$
\begin{array}{cccccccccccc}
 & 1 & + & 1 & + & \cdots & + & 1 & + & 1 & + & 1 \quad + \\
+ & \nu_1^{d_1}-\mu_1^{d_1} & + & \nu_1^{d_1-1}-\mu_1^{d_1-1} & + & \cdots & + & \nu_1^3-\mu_1^3 & + & \nu_1^2-\mu_1^2 & + & \nu_1^1-\mu_1^1 \quad + \\
+ & \nu_2^{d_1}-\mu_2^{d_1} & + & \nu_2^{d_1-1}-\mu_2^{d_1-1} & + & \cdots & + & \nu_2^3-\mu_2^3 & + & \nu_2^2-\mu_2^2 & + & \\
+ & \nu_3^{d_1}-\mu_3^{d_1} & + & \nu_3^{d_1-1}-\mu_3^{d_1-1} & + & \cdots & + & \nu_3^3-\mu_3^3 & + & & & \\
+ & \cdots\cdots\cdots & + & \cdots\cdots\cdots & + & \cdots & + & & & & & \\
+ & \nu_{d_1-1}^{d_1}-\mu_{d_1-1}^{d_1} & + & \nu_{d_1-1}^{d_1-1}-\mu_{d_1-1}^{d_1-1} & + & & & & & & & \\
+ & \nu_{d_1}^{d_1}-\mu_{d_1}^{d_1}. & & & & & & & & & &
\end{array}
$$

But by permuting the order of appearance of ν and μ in each subtraction of every line, this sum becomes:

$$
\begin{array}{cccccccccccc}
 & & 1 & + & \cdots & + & 1 & + & 1 & + & 1 & - \\
-\mu_1^{d_1} & + & \underline{\nu_1^{d_1}-\mu_1^{d_1-1}} & + & \cdots & + & \underline{\nu_1^3-\mu_1^2} & + & \underline{\nu_1^2-\mu_1^1} & + & \nu_1^1 & \\
-\mu_2^{d_1} & + & \underline{\nu_2^{d_1}-\mu_2^{d_1-1}} & + & \cdots & + & \underline{\nu_2^3-\mu_2^2} & + & \nu_2^2 & & & \\
\cdots & + & \cdots\cdots\cdots & + & \cdots & & & & & & & \\
-\mu_{d_1-1}^{d_1} & + & \underline{\nu_{d_1-1}^{d_1}-\mu_{d_1-1}^{d_1-1}} & + & & & & & & & & \\
-\mu_{d_1}^{d_1} & + & \nu_{d_1}^{d_1}, & & & & & & & & &
\end{array}
$$

and then by taking account just of the semi-standard-like inequalities (15) on p. 42 (about the columns of contact between two neighboring blocks), we see that all the pairs that we have underlined above are $\leqslant 0$, whence:

$$
\begin{aligned}
D_1 + D_2 + D_3 + \cdots + D_{d_1-1} + D_{d_1} \leqslant d_1 \cdot 1 + \left(-\mu_1^{d_1} + \nu_1^1\right) + \left(-\mu_2^{d_1} + \nu_2^2\right) + \cdots + \\
+ \left(-\mu_{d_1-1}^{d_1} + \nu_{d_1-1}^{d_1-1}\right) + \left(-\mu_{d_1}^{d_1} + \nu_{d_1}^{d_1}\right).
\end{aligned}
$$

Finally, the strict inequalities $0 < \mu_1^{d_1} < \mu_2^{d_1} < \cdots < \mu_{d_1-1}^{d_1} < \mu_{d_1}^{d_1}$ between the entries of the first column yield trivial majorants:

$$
-\mu_1^{d_1} \leqslant -1, \quad -\mu_2^{d_1} \leqslant -2, \ldots\ldots, -\mu_{d_1-1}^{d-1} \leqslant -(d_1-1), \quad -\mu_{d_1} \leqslant -d_1,
$$

and since all the ν_i^j are $\leqslant \kappa$ anyway, we deduce about any semi-standard Young tableau the following majoration:

$$
\begin{aligned}
\text{total number of distinct } *\text{-ed columns} &\leqslant d_1 + (-1+\kappa) + (-2+\kappa) + \cdots + \\
&\qquad + (-(d_1-1)+\kappa) + (-d_1+\kappa) \\
&\leqslant n + (-1+\kappa) + (-2+\kappa) + \cdots + \\
&\qquad + (-(n-1)+\kappa) + (-n+\kappa) \\
&= n\kappa - \tfrac{n(n-1)}{2}.
\end{aligned}
$$

Lemma. *The total number of pairwise distinct columns in a semi-standard Young tableau of depth* $\leqslant n$ *filled in with integers* $\lambda_i^j \leqslant \kappa$ *is in any case* $\leqslant n\kappa - \frac{n(n-1)}{2}$. □

We now introduce several families of Δ-monomials parametrized by a fixed collection of pairs of columns (having various multiplicities $* \geqslant 1$) that should occupy the left and right extreme positions in blocks of decreasing depths.

Main definition. Let d_1 be a positive integer $\leqslant n$ and let μ_l^i and ν_l^i be integers defined for $i = 1, 2, 3, \dots, d_1 - 1, d_1$ and $1 \leqslant l \leqslant i$ with $\mu_l^i \leqslant \kappa$ and $\nu_l^i \leqslant \kappa$ which satisfy all the following inequalities:

- vertical downward increasing:

$$0 < \mu_1^i < \cdots < \mu_i^i \quad \text{and} \quad 0 < \nu_1^i < \cdots < \nu_i^i \qquad {\scriptstyle (i=1,\dots,d_1)};$$

- weak increasing inside blocks:

$$\mu_l^i \leqslant \nu_l^i \qquad {\scriptstyle (i=1,\dots,d_1;\ 1\leqslant l\leqslant i)};$$

- weak increasing for the contacts between neighboring blocks[14]:

$$\nu_l^{i+1} \leqslant \mu_l^i \qquad {\scriptstyle (i=1,\dots,d_1-1;\ 1\leqslant l\leqslant i)}$$

Then with such data, the family of semi-standard tableaux:

$$\boxed{\mathsf{YT}_{\kappa,m}\big(\mu_l^i, \nu_l^i\big)}$$

is defined to consist of all possible concatenations:

$$\mathsf{block}^{d_1}\big(\mu^{d_1}, \nu^{d_1}\big) \cdots \mathsf{block}^{i}\big(\mu^{i}, \nu^{i}\big) \cdots \mathsf{block}^{1}\big(\mu^{1}, \nu^{1}\big)$$

of *semi-standard* blocks[15] of the form:

$$\mathsf{block}^i\big(\mu^i,\nu^i\big) = \boxed{\begin{bmatrix} \mu_1^i \\ \mu_2^i \\ \cdot \\ \cdot \\ \mu_i^i \end{bmatrix}^{a_{\mu_1^i,\dots,\mu_i^i}} \cdots \begin{bmatrix} \lambda_1^i \\ \lambda_2^i \\ \cdot \\ \cdot \\ \lambda_i^i \end{bmatrix}^{a_{\lambda_1^i,\dots,\lambda_i^i}} \cdots \begin{bmatrix} \nu_1^i \\ \nu_2^i \\ \cdot \\ \cdot \\ \nu_i^i \end{bmatrix}^{a_{\nu_1^i,\dots,\nu_i^i}}},$$

all $*$-ed columns being pairwise distinct and ordered increasingly from left to right, with multiplicities:

$$a_{\mu_1^i,\dots,\mu_i^i} \geqslant 1, \quad \dots \quad a_{\lambda_1^i,\dots,\lambda_i^i} \geqslant 1, \quad \dots \quad a_{\nu_1^i,\dots,\nu_i^i} \geqslant 1$$

which are *all* positive and with the further important constraint that:

$$\begin{aligned} m = {} & \mathrm{weight}\big(\mathsf{block}^{d_1}(\mu^{d_1},\nu^{d_1})\big) + \cdots + \\ & + \mathrm{weight}\big(\mathsf{block}^{i}(\mu^{i},\nu^{i})\big) + \cdots + \mathrm{weight}\big(\mathsf{block}^{1}(\mu^{1},\nu^{1})\big), \end{aligned}$$

---

[14] The most general case where certain block lengths i are missing, so that block lengths sometimes jump for more than one unit, is implicitly also embraced by such a definition, for it suffices, about the indices i of blocks that are thought to be missing, to just prescribe somewhat arbitrarily some integers μ_l^i and ν_l^i that violate the second condition ; the third condition is then supposed to hold for the direct contacts between the really extant neighboring blocks. Since in our later principal considerations, we will not be dealing with semi-standard tableaux having block gaps, it is not necessary to introduce further specific notations here.

[15] By a *semi-standard block* is of course meant a block in which strict increase holds downward along columns, while weak increase holds from left to right along rows.

where according to an expectable, natural definition:

$$\begin{aligned}\mathrm{weight}\big(\mathsf{block}^i(\mu^i,\nu^i)\big) \overset{\mathrm{def}}{=} \big(\mu_1^i+\cdots+\mu_i^i\big)a_{\mu_1^i,\dots,\mu_i^i}+\cdots+\\ +\big(\lambda_1^i+\cdots+\lambda_i^i\big)a_{\lambda_1^i,\dots,\lambda_i^i}+\cdots+\big(\nu_1^i+\cdots+\nu_i^i\big)a_{\nu_1^i,\dots,\nu_i^i}\end{aligned}$$

simply denotes the total number of primes (remember the theorem on p. 22) in the associated Δ-monomial:

$$\big(\Delta_{1,\dots,i}^{\mu_1^i,\dots,\mu_i^i}\big)^{a_{\mu_1^i,\dots,\mu_i^i}}\cdots\big(\Delta_{1,\dots,i}^{\lambda_1^i,\dots,\lambda_i^i}\big)^{a_{\lambda_1^i,\dots,\lambda_i^i}}\cdots\big(\Delta_{1,\dots,i}^{\nu_1^i,\dots,\nu_i^i}\big)^{a_{\nu_1^i,\dots,\nu_i^i}}.$$

In a specific family $\mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i\big)$, the freedom of variation lies: 1) in the choice of some intermediate columns; 2) in the choice of the number of such intermediate columns; 3) in the choice of the positive multiplicities of all the columns.

Lemma. *The collection of all semi-standard Young tableaux* YT *of depth* $\leqslant n$ *filled in with positive integers* $\lambda_i^j\leqslant\kappa$ *whose weight equals* m *is identical to the disjoint union:*

$$\bigcup_{\mu_l^i,\nu_l^i}\mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i\big)$$

of the so-defined families of semi-standard tableaux.

Proof. According to the preceding considerations, an arbitrary semi-standard Young tableau looks like (16) on p. 44, hence belongs to $\mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i\big)$ for some μ_l^i,ν_l^i. Disjointness follows from the fact that the extreme column data $\big(\mu_l^i,\nu_l^i\big)$ are obviously pairwise distinct. $\square$

By what has already been seen, the number of pairwise distinct columns in any $\mathsf{block}^i(\mu^i,\nu^i)$ may well be equal to zero[16] and is always smaller than or equal to:

$$D_i := 1+\textstyle\sum_{l=1}^i\big(\nu_l^i-\mu_l^i\big).$$

In order to fix ideas about the exact number of distinct columns, we shall in addition split each (big) family $\mathsf{YT}_{\kappa,m}(\mu_l^i,\nu_l^i)$ in distinct, finer (sub)families as follows.

For every $i=1,\dots,n$ and for every nonnegative integer:

$$\tau_i\leqslant D_i-1=\textstyle\sum_{l=1}^i\big(\nu_l^i-\mu_l^i\big)$$

less than the maximal possible number of distinct columns inside $\mathsf{block}^i(\mu^i,\nu^i)$ *minus* 1, let us choose a discrete increasing path[17]:

$$\gamma^i\colon\ \big\{0,1,2,\dots,\tau^i\big\}\longrightarrow\textsf{downward increasing columns}\in\{1,\dots,\kappa\}^i$$

[16] This would correspond to the empty block case, *cf.* a preceding footnote.

[17] When a block of depth i is inextant, we set $\tau^i:=-1$ so that the length $1+\tau^i$ of any associated path γ^i equals 0: possible paths γ^i are thus necessarily empty in this case.

from the μ^i-column to the ν^i-column, namely:

$$\begin{bmatrix}\mu_1^i=\gamma_1^i(0)\\ \mu_2^i=\gamma_2^i(0)\\ \cdot\cdot\\ \mu_i^i=\gamma_i^i(0)\end{bmatrix}^* < \begin{bmatrix}\gamma_1^i(1)\\ \gamma_2^i(1)\\ \cdot\\ \gamma_i^i(1)\end{bmatrix}^* <\cdots< \begin{bmatrix}\gamma_1^i(s^i)\\ \gamma_2^i(s^i)\\ \cdot\\ \gamma_i^i(s^i)\end{bmatrix}^* <\cdots< \begin{bmatrix}\gamma_1^i(\tau^i)=\nu_1^i\\ \gamma_2^i(\tau^i)=\nu_2^i\\ \cdot\cdot\\ \gamma_i^i(\tau^i)=\nu_i^i\end{bmatrix}^*,$$

with $s^i=0,1,2,\dots,\tau^i$ denoting the current (discrete) time variable, such that the associated block:

$$\mathsf{block}^i\big(\gamma^i\big):=\boxed{\begin{bmatrix}\gamma_1^i(0)\\ \gamma_2^i(0)\\ \cdot\\ \gamma_i^i(0)\end{bmatrix}^* \begin{bmatrix}\gamma_1^i(1)\\ \gamma_2^i(1)\\ \cdot\\ \gamma_i^i(1)\end{bmatrix}^* \cdots \begin{bmatrix}\gamma_1^i(s^i)\\ \gamma_2^i(s^i)\\ \cdot\\ \gamma_i^i(s^i)\end{bmatrix}^* \cdots \begin{bmatrix}\gamma_1^i(\tau^i)\\ \gamma_2^i(\tau^i)\\ \cdot\\ \gamma_i^i(\tau^i)\end{bmatrix}^*}$$

is semi-standard. Then with such data, the (sub)family of semi-standard tableaux:

$$\boxed{\mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)\big)}$$

is defined to consist of all possible concatenations:

$$\mathsf{block}^n\big(\gamma^n\big)\cdots\mathsf{block}^i\big(\gamma^i\big)\cdots\mathsf{block}^1\big(\gamma^1\big)$$

of the above specific blocks, with $*$-multiplicities:

$$a_{\gamma_1^i(0),\dots,\gamma_i^i(0)}\geqslant 1,\qquad \dots\qquad a_{\gamma_1^i(s^i),\dots,\gamma_i^i(s^i)}\geqslant 1,\qquad \dots\qquad a_{\gamma_1^i(\tau^i),\dots,\gamma_i^i(\tau^i)}\geqslant 1$$

which are *all* positive, and with the further constraint, similar as before, that:

$$m=\mathrm{weight}\big(\mathsf{block}^n(\gamma^n)\big)+\cdots+\mathrm{weight}\big(\mathsf{block}^i(\gamma^i)\big)+\cdots+\mathrm{weight}\big(\mathsf{block}^1(\gamma^1)\big).$$

Here of course, the weight of a general single column, namely having with multiplicity 1, equals:

$$\gamma_1^i(s^i)+\gamma_2^i(s^i)+\cdots+\gamma_i^i(s^i),$$

hence the $*$-ed column has weight:

$$\big[\gamma_1^i(s^i)+\gamma_2^i(s^i)+\cdots+\gamma_i^i(s^i)\big]\,a_{\gamma_1^i(s^i),\dots,\gamma_i^i(s^i)}.$$

From now on, we shall denote the multiplicity of a general $*$-ed column shortly by $a_{s^i}^i$, instead of $a_{\gamma_1^i(s^i),\dots,\gamma_i^i(s^i)}$. The weight homogeneity condition therefore reads:

$$m=\sum_{i=1}^n\sum_{0\leqslant s^i\leqslant\tau^i}\big[\gamma_1^i(s^i)+\cdots+\gamma_i^i(s^i)\big]a_{s^i}^i. \tag{18}$$

In a specific family $\mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)\big)$, the freedom of variation now lies only in the multiplicities, since all the pairwise distinct $*$-ed columns are fully prescribed in it. Notice that as m is supposed to be quite large[18] in comparison to n and κ, then for any choice of pairwise distinct column data $\big(\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)\big)$, the column weights $\big[\gamma_1^i(s^i)+\cdots+\gamma_i^i(s^i)\big]$ being fixed and

[18] We will eventually let $m\to\infty$, similarly as in the Euler-Poincaré characteristic of $\mathscr{E}_{\kappa,m}^{GG}T_X^*$.

finite, there is still much freedom for the multiplicities to fulfill the homogeneity condition in question. We will in fact realize in a while that the number of semi-standard Young tableaux of weight m in any family $\mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)\big)$ is an $\mathsf{O}_{n,\kappa}\big(m^{D-1}\big)$, where $D=\sum_{i=1}^n(1+\tau^i)$ as before is the total number of pairwise distinct columns.

By construction, it is clear that the union of the (sub)families $\mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)\big)$ fills the previously introduced larger family:

$$\mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i\big)=\bigcup_{\tau^i,\gamma^i(s^i)}\mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)\big).$$

Lemma. *The collection of all semi-standard Young tableaux* YT *of depth* $\leqslant n$ *filled in with positive integers* $\lambda_i^j\leqslant\kappa$ *whose weight equals* m *is identical to the* disjoint *union:*

$$\bigcup_{\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)}\mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)\big)$$

of the so-defined families of semi-standard tableaux. Furthermore, the number of possible such families $\mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)\big)$ *is smaller than or equal to the (nonoptimal) constant:*

$$\prod_{i=1}^n\big(1+\tfrac{\kappa!}{(\kappa-i)!\,i!}\big)^{1+i(\kappa-i)}=\mathsf{Constant}_{n,\kappa},$$

independently of m*.*

Proof. Disjointness (not yet argued) of subfamilies inside a family $\mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i\big)$ comes from the fact that any collection of paths $(\gamma^1,\gamma^2,\dots,\gamma^n)$ prescribes all the mutually distinct $*$-ed columns that are extant, their multiplicities being all $\geqslant 1$.

In a block of depth i, a single $*$-ed column is either empty, or it consists of i numbers $\lambda_1,\dots,\lambda_i$ chosen without repetition in $\{1,2,\dots,\kappa\}$ and ordered afterward increasingly. So the number of possible such columns (including the empty one) is equal to $1+\frac{\kappa!}{(\kappa-i)!\,i!}$. Since all $*$-ed columns are pairwise distinct, the maximal number of $*$-ed columns that one may put in a semi-standard block of depth i will be attained for the blocks having the following two extreme columns, which are the farthest ones for the ordering between columns of depth i:

$$\boxed{\begin{bmatrix}1\\2\\\cdot\\i-1\\i\end{bmatrix}^*\cdots\begin{bmatrix}\kappa-i+1\\\kappa-i+2\\\cdot\\\kappa-1\\\kappa\end{bmatrix}^*}.$$

It follows from (17) on p. 44 that one may put at most:

(19) $$1+(\kappa-i+1-1)+(\kappa-i+2-2)+\cdots+\kappa-i=1+i(\kappa-i)$$

pairwise distinct $*$-ed columns in between so as to constitute a semi-standard block. Without optimality, we then majorate the number of possible semi-standard blocks of depth i (including the empty one) simply by the number $1+\frac{\kappa!}{(\kappa-i)!\,i!}$ of possible $*$-ed columns raised to a power equal to the maximal number $1+i(\kappa-i)$ of pairwise distinct such $*$-ed columns. What matters for the sequel is only that the obtained majorant is independent of m. $\square$

In summary, here is how we constitute our refined view of an arbitrary semi-standard Young tableau: the data $(\mu_l^i,\nu_l^i)_{1\leqslant l\leqslant i}$, subjected to the natural inequalities of the Main definition on p. 46, prescribe the left and right extreme $*$-ed column in all blocks of depth $i=1,2,\dots,n$ (with possible block gaps); τ^i+1 is the number of pairwise distinct $*$-ed columns in the i-th block, and these columns are $\gamma^i(0),\dots,\gamma^i(\tau^i)$; all $*$-multiplicities of these columns are $\geqslant 1$.

Asymptotically negligible families of Δ-monomials. By definition, for each semi-standard Young tableau YT belonging to a fixed family:

$$\mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)\big),$$

the number D of pairwise distinct $*$-ed columns is equal to the sum of the lengths of the paths between two extreme $*$-ed columns in every block[19]:

$$D=(1+\tau^1)+\cdots+(1+\tau^i)+\cdots+(1+\tau^n).$$

However, the common horizontal length of each of the i rows in the block:

$$\mathsf{block}^i(\gamma^i)=\boxed{\begin{bmatrix}\gamma_1^i(0)\\ \gamma_2^i(0)\\ \cdot\\ \gamma_i^i(0)\end{bmatrix}^{a_0^i}\begin{bmatrix}\gamma_1^i(1)\\ \gamma_2^i(1)\\ \cdot\\ \gamma_i^i(1)\end{bmatrix}^{a_1^i}\cdots\begin{bmatrix}\gamma_1^i(s^i)\\ \gamma_2^i(s^i)\\ \cdot\\ \gamma_i^i(s^i)\end{bmatrix}^{a_{s^i}^i}\cdots\begin{bmatrix}\gamma_1^i(\tau^i)\\ \gamma_2^i(\tau^i)\\ \cdot\\ \gamma_i^i(\tau^i)\end{bmatrix}^{a_{\tau^i}^i}}$$

depends visibly on the multiplicities, and is equal to:

$$a_0^i+a_1^i+\cdots+a_{s^i}^i+\cdots+a_{\tau^i}^i=\sum_{0\leqslant s^i\leqslant\tau^i}a_{s^i}^i.$$

[19] By the preceding convention, inextant blocks contribute with 0 to this sum, *e.g.* all blocks of depths $d_1+1,\dots,n$ when the depth d_1 of the tableau is $<n$.

It follows that the lengths $\ell_1, \ell_2, \dots, \ell_{n-1}, \ell_n$ of the first, second, ..., $(n-1)$-th and n-th horizontal lines in the semi-standard Young tableau YT are equal to[20]:

$$\begin{aligned}
\ell_1 &= \sum_{0\leqslant s^n\leqslant\tau^n} a^n_{s^n} + \sum_{0\leqslant s^{n-1}\leqslant\tau^{n-1}} a^{n-1}_{s^{n-1}} + \cdots + \sum_{0\leqslant s^2\leqslant\tau^2} a^2_{s^2} + \sum_{0\leqslant s^1\leqslant\tau^1} a^1_{s^1}\\
\ell_2 &= \sum_{0\leqslant s^n\leqslant\tau^n} a^n_{s^n} + \sum_{0\leqslant s^{n-1}\leqslant\tau^{n-1}} a^{n-1}_{s^{n-1}} + \cdots + \sum_{0\leqslant s^2\leqslant\tau^2} a^2_{s^2}\\
\cdots &= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\ell_{n-1} &= \sum_{0\leqslant s^n\leqslant\tau^n} a^n_{s^n} + \sum_{0\leqslant s^{n-1}\leqslant\tau^{n-1}} a^{n-1}_{s^{n-1}}\\
\ell_n &= \sum_{0\leqslant s^n\leqslant\tau^n} a^n_{s^n}.
\end{aligned}$$

As we already said in Section 6 above, it appears *a posteriori* more adequate to write everything in terms of the differences:

$$\ell_1-\ell_2 = \sum_{0\leqslant s^1\leqslant\tau^1} a^1_{s^1}, \ \ \dots, \ \ \ell_{n-1}-\ell_n = \sum_{0\leqslant s^{n-1}\leqslant\tau^{n-1}} a^{n-1}_{s^{n-1}}, \qquad \ell_n = \sum_{0\leqslant s^n\leqslant\tau^n} a^n_{s^n}$$

of lengths of lines, which are nothing but row lengths of blocks.

Proposition. *Fix μ^i_l, ν^i_l, τ^i and $\gamma^i(s^i)$. If $\alpha'_1, \dots, \alpha'_{n-1}, \alpha'_n$ are arbitrary nonnegative integers satisfying:*

$$\alpha'_1 + \cdots + \alpha'_{n-1} + \alpha'_n \leqslant \tfrac{n(n+1)}{2},$$

then there exists a positive quantity $\mathsf{Constant}_{n,\kappa} > 0$ depending on n and on κ which is independent of m such that:

$$\begin{aligned}
\sum_{\mathsf{YT}\in\mathsf{YT}_{\kappa,m}(\mu^i_l,\nu^i_l,\tau^i,\gamma^i(s^i))} &\big(\ell_1(\mathsf{YT})-\ell_2(\mathsf{YT})\big)^{\alpha'_1}\cdots\big(\ell_{n-1}(\mathsf{YT})-\ell_n(\mathsf{YT})\big)^{\alpha'_{n-1}}\big(\ell_n(\mathsf{YT})\big)^{\alpha'_n} \leqslant\\
&\leqslant \mathsf{Constant}_{n,\kappa}\cdot m^{\alpha'_1+\cdots+\alpha'_{n-1}+\alpha'_n}\cdot m^{D-1},
\end{aligned}$$

where $D = \sum_{i=1}^n (1+\tau^i)$ is the common number of pairwise distinct $*$-ed columns shared by all Young tableaux $\mathsf{YT}\in\mathsf{YT}_{\kappa,m}\big(\mu^i_l,\nu^i_l,\tau^i,\gamma^i(s^i)\big)$.

Proof. Substituting the values $\ell_i - \ell_{i+1} = \sum_{0\leqslant s^i\leqslant\tau^i} a^i_{s^i}$ in the monomial:

$$\big(\ell_1-\ell_2\big)^{\alpha'_1}\cdots\big(\ell_{n-1}-\ell_n\big)^{\alpha'_{n-1}}\big(\ell_n\big)^{\alpha'_n},$$

and expanding the result, we may majorate:

$$\begin{aligned}
&\big(\ell_1-\ell_2\big)^{\alpha'_1}\cdots\big(\ell_{n-1}-\ell_n\big)^{\alpha'_{n-1}}\big(\ell_n\big)^{\alpha'_n} \leqslant\\
&\leqslant \mathsf{Constant}_{\tau^1,\dots,\tau^n}\cdot \sum_{\sum\beta^1_{s^1}+\cdots+\sum\beta^n_{s^n}=\alpha'_1+\cdots+\alpha'_{n-1}+\alpha'_n} \bigg(\prod_{i=1}^n\prod_{0\leqslant s^i\leqslant\tau^i}\big(a^i_{s^i}\big)^{\beta^i_{s^i}}\bigg).
\end{aligned}$$

[20] Sums $\sum_{0\leqslant s^i\leqslant\tau^i} a^i_{s^i}$ for which $\tau^i = -1$ (inextant blocks) are naturally thought to be inextant.

Since according to (19) on p. 49 above, the $\tau^i \leqslant i(\kappa - i) \leqslant n\kappa$ are majorated in terms of n and κ, we have:

$$\mathsf{Constant}_{\tau^1,\dots,\tau^n} \leqslant \mathsf{Constant}_{n,\kappa}.$$

Moreover, since $\alpha_1' + \cdots + \alpha_{n-1}' + \alpha_n' \leqslant \frac{n(n+1)}{2}$, the number of terms in the sum:

$$\sum_{\sum \beta_{s^1}^1 + \cdots + \sum \beta_{s^n}^n = \alpha_1' + \cdots + \alpha_{n-1}' + \alpha_n'} (\bullet)$$

is also $\leqslant \mathsf{Constant}_{n,\kappa}$. Consequently, in order to prove the proposition, it suffices to majorate by the same claimed majorant:

$$\mathsf{Constant}_{n,\kappa} \cdot m^{\alpha_1' + \cdots + \alpha_{n-1}' + \alpha_n'} \cdot m^{D-1}$$

every single sum of the form:

$$\sum_{\mathsf{YT} \in \mathsf{YT}_{\kappa,m}\left(\mu_l^i, \nu_l^i, \tau^i, \gamma^i(s^i)\right)} \bigg(\prod_{0 \leqslant s^1 \leqslant \tau^1} \left(a_{s^1}^1\right)^{\beta_{s^1}^1} \cdots \prod_{0 \leqslant s^n \leqslant \tau^n} \left(a_{s^n}^n\right)^{\beta_{s^n}^n} \bigg),$$

where the exponents $\beta_{s^i}^i$, $i = 1, \dots, n$, $0 \leqslant s^i \leqslant \tau^i$, are now fixed and subjected to the same property that their sum equals:

$$\sum_{i=1}^n \sum_{0 \leqslant s^i \leqslant \tau^i} \beta_{s^i}^i = \alpha_1' + \cdots + \alpha_{n-1}' + \alpha_n'.$$

Recall that Young tableaux in the family $\mathsf{YT}_{\kappa,m}\left(\mu_l^i, \nu_l^i, \tau^i, \gamma^i(s^i)\right)$ have fixed set of pairwise distinct columns, and that the freedom only lies in the multiplicities $a_{s^i}^i \geqslant 1$, $i = 1, \dots, n$, $0 \leqslant s^i \leqslant \tau^i$, of the columns. The considered sum:

$$\sum_{\mathsf{YT} \in \mathsf{YT}_{\kappa,m}\left(\mu_l^i, \nu_l^i, \tau^i, \gamma^i(s^i)\right)} (\bullet)$$

coincides therefore with the sum:

$$\sum_{\sum_{i=1}^n \sum_{0 \leqslant s^i \leqslant \tau^i} \left[\gamma_1^i(s^i) + \cdots + \gamma_i^i(s^i)\right] a_{s^i}^i = m} (\bullet),$$

which takes precisely account of the homogeneity constraint (18) on p. 48. Let us now set:

$$b_{s^i}^i := \left[\gamma_1^i(s^i) + \cdots + \gamma_i^i(s^i)\right] \, a_{s^i}^i,$$

whence $a_{s^i}^i \leqslant b_{s^i}^i$ always[21], so that the sum in question now writes:

$$\sum_{\sum b_{s^1}^1 + \cdots + \sum b_{s^n}^n = m} \bigg(\prod_{i=1}^n \prod_{0 \leqslant s^i \leqslant \tau^i} \left(a_{s^i}^i\right)^{\beta_{s^i}^i} \bigg)$$
$$\leqslant \sum_{\sum b_{s^1}^1 + \cdots + \sum b_{s^n}^n = m} \bigg(\prod_{i=1}^n \prod_{0 \leqslant s^i \leqslant \tau^i} \left(b_{s^i}^i\right)^{\beta_{s^i}^i} \bigg).$$

[21] Even in the case where the block of depth i is inextant.

The number of nonzero variables $b_{s^i}^i \in \mathbb{N}$ here is the same, equal to D, as the number of nonzero exponents $a_{s^i}^i$. The conclusion now follows from the elementary general inequality:

$$\sum_{\substack{b_1+\cdots+b_D=m\\ b_1\geqslant 1,\dots,b_D\geqslant 1}} b_1^{\beta_1}\cdots b_D^{\beta_D} \leqslant \mathsf{Constant}_D \cdot m^{\beta_1+\cdots+\beta_D}\cdot m^{D-1},$$

that can be established by approximating the sum by a Riemann integral; of course, $\mathsf{Constant}_D \leqslant \mathsf{Constant}_{n,\kappa}$. □

From this proposition, we will deduce a few corollaries. Firstly, as announced earlier on at the end of Section 6, we have:

Corollary. *Let $\alpha_1',\dots,\alpha_{n-1}',\alpha_n'$ be nonnegative integers satisfying:*

$$\alpha_1'+\cdots+\alpha_{n-1}'+\alpha_n' \leqslant \tfrac{n(n+1)}{2}-1.$$

Then the following majoration holds for the summation over all *semi-standard Young tableaux of weight m:*

$$\begin{aligned}\sum_{\substack{\mathsf{YT}\text{ semi-standard}\\ \mathrm{weight}(\mathsf{YT})=m}} \big(\ell_1(\mathsf{YT})-\ell_2(\mathsf{YT})\big)^{\alpha_1'}\cdots\big(\ell_{n-1}(\mathsf{YT})-\ell_n(\mathsf{YT})\big)^{\alpha_{n-1}'}\big(\ell_n(\mathsf{YT})\big)^{\alpha_n'} &\leqslant\\ \leqslant \mathsf{Constant}_{n,\kappa}\cdot m^{\alpha_1'+\cdots+\alpha_{n-1}'+\alpha_n'}\cdot m^{n\kappa-\frac{n(n-1)}{2}}&\\ \leqslant \mathsf{Constant}_{n,\kappa}\cdot m^{(\kappa+1)n-2}.&\end{aligned}$$

Proof. According to the lemma on p. 49:

$$\sum_{\substack{\mathsf{YT}\text{ semi-standard}\\ \mathrm{weight}(\mathsf{YT})=m}} (\,\bullet\,) = \sum_{\mu_l^i,\,\nu_l^i}\sum_{\tau_i}\sum_{\gamma^i(s^i)}\sum_{\mathsf{YT}\in\mathsf{YT}(\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i))}(\,\bullet\,),$$

and furthermore, the number of terms in the three first sums of the right-hand side is $\leqslant \mathsf{Constant}_{n,\kappa}$. It suffices then to apply the proposition which controls each fourth sum, reminding from the lemma on p. 45 that each $D=\sum_{i=1}^n(1+\tau^i)$ is in any case $\leqslant n\kappa-\frac{n(n-1)}{2}$. □

Secondly, from the simple inequality (13) on p. 38, we immediately deduce:

Corollary. *If $\alpha_1,\dots,\alpha_n$ are any nonnegative integers satisfying $\alpha_1+\cdots+\alpha_n\leqslant \frac{n(n+1)}{2}-1$, then:*

$$\begin{aligned}\sum_{\substack{\mathsf{YT}\text{ semi-standard}\\ \mathrm{weight}(\mathsf{YT})=m}} \big(\ell_1(\mathsf{YT})\big)^{\alpha_1}\cdots\big(\ell_n(\mathsf{YT})\big)^{\alpha_n} &\leqslant \mathsf{Constant}_{n,\kappa}\cdot m^{\alpha_1+\cdots+\alpha_n}\cdot m^{n\kappa-\frac{n(n-1)}{2}}\\ &\leqslant \mathsf{Constant}_{n,\kappa}\cdot m^{(\kappa+1)n-2}.\end{aligned}$$

Lastly, we now introduce a certain collection of semi-standard Young tableaux the contribution of which appears to also fall in the remainder

$\mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big)$: we gather all the ones for which the number of pairwise distinct columns is (strictly) less than the maximal possible number $n\kappa - \frac{n(n-1)}{2}$:

$$\boxed{\mathsf{NGYT}_{\kappa,m} := \bigcup_{\substack{\mu_l^i, \nu_l^i, \tau^i, \gamma^i(s^i) \\ \sum_{i=1}^n (1+\tau^i) \leqslant n\kappa - \frac{n(n-1)}{2} - 1}} \mathsf{YT}_{\kappa,m}\big(\mu_l^i, \nu_l^i, \tau^i, \gamma^i(s^i)\big)}.$$

The number of appearing such families $\mathsf{YT}_{\kappa,m}\big(\mu_l^i, \nu_l^i, \tau^i, \gamma^i(s^i)\big)$ is of course less than the majorant:

$$\prod_{i=1}^n \big(1 + \tfrac{\kappa!}{(\kappa-i)!\, i!}\big)^{1+i(\kappa-i)} \equiv \mathsf{Constant}_{n,\kappa}$$

provided by the lemma on p. 49 for the total number of all families.

Lemma. *For any integers $\alpha_1', \dots, \alpha_{n-1}', \alpha_n'$ whose sum equals $\frac{n(n+1)}{2}$, the contribution of:*

$$\sum_{\mathsf{YT} \in \mathsf{NGYT}_{\kappa,m}} \big(\ell_1(\mathsf{YT}) - \ell_2(\mathsf{YT})\big)^{\alpha_1'} \cdots \big(\ell_{n-1}(\mathsf{YT}) - \ell_n(\mathsf{YT})\big)^{\alpha_{n-1}'} \big(\ell_n(\mathsf{YT})\big)^{\alpha_n'} \leqslant$$
$$\leqslant \mathsf{Constant}_{n,\kappa} \cdot m^{(\kappa+1)n-2}$$

is asymptotically negligible in comparison to the dominant power $m^{(\kappa+1)n-1}$.

Proof. By what has been just seen, it suffices to verify that such a majoration holds for a sum $\sum_{\mathsf{YT} \in (\bullet)}$ running over a single family $\mathsf{YT}_{\kappa,m}\big(\mu_l^i, \nu_l^i, \tau^i, \gamma^i(s^i)\big)$ contained in $\mathsf{NGYT}_{\kappa,m}$, and this was already achieved by the proposition on p. 51 above, since $\frac{n(n+1)}{2} + D - 1 \leqslant (\kappa+1)n - 2$ always when $D \leqslant n\kappa - \frac{n(n-1)}{2} - 1$. □

§8. Maximal length families of semi-standard Young tableaux

Relevant families of Δ-monomials. In the remainder of the paper, we shall now consider only exponents α_i' satisfying $\alpha_1' + \cdots + \alpha_{n-1}' + \alpha_n' = \frac{n(n+1)}{2}$. From the proposition just proved, we deduce that the complete sum:

(20)
$$\sum_{\substack{\mathsf{YT}\ \text{semi-standard} \\ \text{weight}(\mathsf{YT}) = m}} \big(\ell_1(\mathsf{YT}) - \ell_2(\mathsf{YT})\big)^{\alpha_1'} \cdots \big(\ell_{n-1}(\mathsf{YT}) - \ell_n(\mathsf{YT})\big)^{\alpha_{n-1}'} \big(\ell_n(\mathsf{YT})\big)^{\alpha_n'}$$

splits up as a negligible sum plus a relevant sum that we should now study:

$$\sum_{\mathsf{YT} \in \mathsf{NGYT}_{\kappa,m}} + \sum_{\mathsf{YT} \notin \mathsf{NGYT}_{\kappa,m}}.$$

Hence, what remains to be studied is the collection of all families of semi-standard Young tableaux:

$$\boxed{\mathsf{YT}_{\kappa,m}^{\max} := \bigcup_{\substack{\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)\\ \sum_{i=1}^n(1+\tau^i)=n\kappa-\frac{n(n-1)}{2}}} \mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)\big)}$$

for which the number of pairwise distinct columns is maximal, equal to $n\kappa - \frac{n(n-1)}{2}$. The following statement describes them in great details.

Proposition. *The number D of pairwise distinct columns in any semi-standard Young tableau written as in* (16) *on p. 44 is in any case $\leqslant n\kappa - \frac{n(n-1)}{2}$. Furthermore, a given semi-standard Young tableau reaches the maximal number:*

$$D = n\kappa - \tfrac{n(n-1)}{2}$$

of pairwise distinct columns if and only if all the following conditions are fulfilled:

- *the depth of the tableau is maximal: $d_1 = n$;*
- *nonvoid blocks of any depth $i = 1, 2, 3, \dots, n-1, n$ are all extant, so that the number of nonvoid blocks is maximal, equal to n;*
- *the leftmost $*$-ed column of the tableau corresponds to the n-dimensional Wronskian $\Delta^{1,2,3,\dots,n-1,n}_{1,2,3,\dots,n-1,n}$, reproduced a certain number $* \geqslant 1$ of times;*
- *the bottom-right entry of every block is maximal:*

$$\nu_1^1 = \nu_2^2 = \nu_3^3 = \cdots = \nu_{n-1}^{n-1} = \nu_n^n = \kappa;$$

$$\underbrace{\boxed{\begin{bmatrix}1\\2\\3\\\cdot\\\cdot\\n-1\\n\end{bmatrix}^* \cdots \begin{bmatrix}\mu_1^{n-1}\\\mu_2^{n-1}\\\mu_3^{n-1}\\\cdot\\\cdot\\\mu_{n-1}^{n-1}\\\kappa\end{bmatrix}^*}}_{1+\tau^n} \underbrace{\boxed{\begin{bmatrix}\mu_1^{n-1}\\\mu_2^{n-1}\\\mu_3^{n-1}\\\cdot\\\cdot\\\mu_{n-1}^{n-1}\end{bmatrix}^* \cdots \begin{bmatrix}\mu_1^{n-2}\\\mu_2^{n-2}\\\mu_3^{n-2}\\\cdot\\\cdot\\\kappa\end{bmatrix}^*}}_{1+\tau^{n-1}} \cdots \underbrace{\boxed{\begin{bmatrix}\mu_1^3\\\mu_2^3\\\mu_3^3\end{bmatrix}^* \cdots \begin{bmatrix}\mu_1^2\\\mu_2^2\\\kappa\end{bmatrix}^*}}_{1+\tau^3} \underbrace{\boxed{\begin{bmatrix}\mu_1^2\\\mu_2^2\end{bmatrix}^* \cdots \begin{bmatrix}\mu_1^1\\\kappa\end{bmatrix}^*}}_{1+\tau^2} \underbrace{\boxed{\big[\mu_1^1\big]^* \cdots \big[\kappa\big]^*}}_{1+\tau^1}.$$

- *the border column entries (excepting the last one, equal to κ, of the longest column) of any pair of neighboring blocks are equal:*

$$\cdots \boxed{\begin{bmatrix}\mu_1^{i+1}\\\mu_2^{i+1}\\\cdot\\\cdot\\\mu_{i-1}^{i+1}\\\mu_i^{i+1}\\\mu_{i+1}^{i+1}\end{bmatrix}^* \cdots \begin{bmatrix}\nu_1^{i+1}=\mu_1^i\\\nu_2^{i+1}=\mu_2^i\\\cdot\\\cdot\\\nu_{i-1}^{i+1}=\mu_{i-1}^i\\\nu_i^{i+1}=\mu_i^i\\\kappa\end{bmatrix}^*} \boxed{\begin{bmatrix}\mu_1^{i}\\\mu_2^{i}\\\cdot\\\cdot\\\mu_{i-1}^{i}\\\mu_i^{i}\end{bmatrix}^* \cdots \begin{bmatrix}\nu_1^{i}=\mu_1^{i-1}\\\nu_2^{i}=\mu_2^{i-1}\\\cdot\\\cdot\\\nu_{i-1}^{i}=\mu_{i-1}^{i-1}\\\kappa\end{bmatrix}^*} \boxed{\begin{bmatrix}\mu_1^{i-1}\\\mu_2^{i-1}\\\cdot\\\cdot\\\mu_{i-1}^{i-1}\end{bmatrix}^* \cdots \begin{bmatrix}\nu_1^{i-1}=\mu_1^{i-2}\\\nu_2^{i-1}=\mu_2^{i-2}\\\cdot\\\cdot\\\kappa\end{bmatrix}^*} \cdots,$$

- *the number of pairwise distinct columns in each block of depth i, for $i = 1, 2, 3, \dots, n-1, n$ is maximal*[22]*, equal to:*

$$\begin{aligned}1+\tau^i &:= 1 + (\mu_1^{i-1} - \mu_1^i) + (\mu_2^{i-1} - \mu_2^i) + \cdots + (\mu_{i-1}^{i-1} - \mu_{i-1}^i) + (\kappa - \mu_i^i) \\ &= 1 + \kappa + \sum_{l=1}^{i-1} \mu_l^{i-1} - \sum_{l=1}^{i} \mu_l^i,\end{aligned}$$

so that the total number of pairwise distinct columns is accordingly indeed equal to:

$$\begin{aligned}(1+\tau^1)+(1+\tau^2)+(1+\tau^3)+\cdots+(1+\tau^{n-1})+(1+\tau^n) &= n + n\kappa - \sum_{l=1}^{n} \mu_l^n \\ &= n\kappa - \tfrac{n(n-1)}{2}.\end{aligned}$$

Proof. The majorant $n\kappa - \frac{n(n-1)}{2}$ has already been obtained above, before the introduction of families of Young tableaux. The remaining statements follow by thinking once again about what has already been seen above. □

So the families of semi-standard Young tableaux having maximal number $n\kappa - \frac{n(n-1)}{2}$ of pairwise distinct columns is parameterized by all the collections of integers μ_i^j satisfying the inequalities:

$$\begin{aligned}&1 \leqslant \mu_1^1 < \kappa \\ &1 \leqslant \mu_1^2 < \mu_2^2 < \kappa \\ &1 \leqslant \mu_1^3 < \mu_2^3 < \mu_3^3 < \kappa \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &1 \leqslant \mu_1^{n-1} < \mu_2^{n-1} < \mu_3^{n-1} < \cdots < \mu_{n-1}^{n-1} < \kappa \\ &1 \leqslant \mu_1^n < \mu_2^n < \mu_3^n < \cdots < \mu_{n-1}^n < \mu_n^n,\end{aligned} \tag{21}$$

together with the further semi-standard-like inequalities[23]:

$$\mu_l^l \geqslant \mu_l^{l+1} \geqslant \cdots \geqslant \mu_l^{n-1} (\geqslant l). \tag{22}$$

For brevity, let us write as:

$$\boxed{\mu_l^i \in \nabla_{n,\kappa}}$$

the condition that the μ_l^i satisfy the two sets of inequalities (21) and (22). For any such choice of $\mu_l^i \in \nabla_{n,\kappa}$, we shall denote by:

$$\boxed{\mathsf{YT}_{\kappa,m}^{\max}\big(\mu_l^i\big)}$$

---

[22] By convention, we shall also call $\mu_1^n, \mu_2^n, \mu_3^n, \dots, \mu_{n-1}^n, \mu_n^n$ the entries $1, 2, 3, \dots, n-1, n$ of the leftmost $*$-ed column.

[23] Diagramatically, this second set of inequalities reads as saying that *vertically* in each column of the first array (21) of inequalities, the integers μ_l^i are weakly decreasing. In particular, $\kappa \geqslant \mu_n^n = n$, as was assumed throughout earlier on.

the family of semi-standard Young tableaux which consist of all possible concatenations:

$$\mathsf{block}^n\big(\gamma^n\big)\cdots\mathsf{block}^i\big(\gamma^i\big)\cdots\mathsf{block}^1\big(\gamma^1\big)$$

of pathed blocks of the form:

$$\mathsf{block}^i\big(\gamma^i\big) := \boxed{\begin{bmatrix} \mu_1^i=\gamma_1^i(0) \\ \mu_2^i=\gamma_2^i(0) \\ \cdot\cdot \\ \mu_{i-1}^i=\gamma_{i-1}^i(0) \\ \mu_i^i=\gamma_i^i(0) \end{bmatrix}^{a_0^i} \begin{bmatrix} \gamma_1^i(1) \\ \gamma_2^i(1) \\ \cdot \\ \gamma_{i-1}^i(1) \\ \gamma_i^i(1) \end{bmatrix}^{a_1^i} \cdots \begin{bmatrix} \gamma_1^i(s^i) \\ \gamma_2^i(s^i) \\ \cdot \\ \gamma_{i-1}^i(s^i) \\ \gamma_i^i(s^i) \end{bmatrix}^{a_{s^i}^i} \cdots \begin{bmatrix} \gamma_1^i(\tau^i)=\mu_1^{i-1} \\ \gamma_2^i(\tau^i)=\mu_2^{i-1} \\ \cdot\cdot \\ \gamma_{i-1}^i(\tau^i)=\mu_{i-1}^{i-1} \\ \gamma_i^i(\tau^i)=\kappa \end{bmatrix}^{a_{\tau^i}^i}},$$

where the lengths $1+\tau^i$ of paths are maximal equal to:

$$1+\tau^i = 1+\kappa+\sum_{l=1}^{i-1}\mu_l^{i-1}-\sum_{l=1}^{i}\mu_l^i,$$

so that between two successive $*$-ed columns:

$$\begin{bmatrix} \gamma_1^i(s^i) \\ \gamma_2^i(s^i) \\ \cdot \\ \gamma_i^i(s^i) \end{bmatrix}^* < \begin{bmatrix} \gamma_1^i(s^i+1) \\ \gamma_2^i(s^i+1) \\ \cdot\cdot \\ \gamma_i^i(s^i+1) \end{bmatrix}^*$$

one has the semi-standard inequalities $\gamma_l^i(s^i)\leqslant\gamma_l^i(s^i+1)$ for $l=1,2,\dots,i$ but the jump is smallest possible, namely $\gamma_l^i(s^i+1)=\gamma_l^i(s^i)$ for all l except only one l_0 for which:

$$\gamma_{l_0}^i(s^i+1) = 1+\gamma_{l_0}^i(s^i).$$

Such paths all of which jumps are unit will be called *tight paths*. With all these notations, the initial complete sum (20) to be studied now writes:

$$\sum_{\substack{\mathsf{YT\ semi-standard} \\ \mathsf{weight(YT)}=m}}(\bullet) = \sum_{\mathsf{YT}\in\mathsf{NGYT}_{\kappa,m}}(\bullet)+\sum_{\mu_l^i\in\nabla_{n,\kappa}}\ \sum_{\mathsf{YT}\in\mathsf{YT}^{\max}_{\kappa,m}(\mu_l^i)}(\bullet),$$

the first sum being negligible, in the sense that:

(23)

$$\boxed{\begin{aligned} &\sum_{\substack{\mathsf{YT\ semi-standard} \\ \mathsf{weight(YT)}=m}}\big(\ell_1(\mathsf{YT})-\ell_2(\mathsf{YT})\big)^{\alpha_1'}\cdots\big(\ell_{n-1}(\mathsf{YT})-\ell_n(\mathsf{YT})\big)^{\alpha_{n-1}'}\big(\ell_n(\mathsf{YT})\big)^{\alpha_n'} = \\ &= \mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big)+ \\ &+\sum_{\mu_l^i\in\nabla_{n,\kappa}}\ \sum_{\mathsf{YT}\in\mathsf{YT}^{\max}_{\kappa,m}(\mu_l^i)}\big(\ell_1(\mathsf{YT})-\ell_2(\mathsf{YT})\big)^{\alpha_1'}\cdots\big(\ell_{n-1}(\mathsf{YT})-\ell_n(\mathsf{YT})\big)^{\alpha_{n-1}'}\big(\ell_n(\mathsf{YT})\big)^{\alpha_n'} \end{aligned}}.$$

Grouping sums. Let us therefore fix $\mu_l^i\in\nabla_{n,\kappa}$, let us keep aside (and in mind) the first summation $\sum_{\mu_l^i\in\nabla_{n,\kappa}}(\bullet)$, and let us consider only the second summation:

$$\sum_{\mathsf{YT}\in\mathsf{YT}^{\max}_{\kappa,m}(\mu_l^i)}\big(\ell_1(\mathsf{YT})-\ell_2(\mathsf{YT})\big)^{\alpha_1'}\cdots\big(\ell_{n-1}(\mathsf{YT})-\ell_n(\mathsf{YT})\big)^{\alpha_{n-1}'}\big(\ell_n(\mathsf{YT})\big)^{\alpha_n'}.$$

With $\tau^i := \kappa + \sum_{l=1}^{i-1} \mu_l^{i-1} - \sum_{l=1}^{i} \mu_l^i$, this apparently compact sum identifies in fact with the multiple sums:

$$\sum_{\gamma^1(s^1)} \cdots \sum_{\gamma^n(s^n)} \sum_{a_{s^1}^1} \cdots \sum_{a_{s^n}^n} (\bullet),$$

the paths $\gamma^i(s^i)$ being all tight and the multiplicities $a_{s^i}^i$ being constrained only by the weight condition:

$$\sum_{i=1}^{n} \sum_{0 \leqslant s^i \leqslant \tau^i} \big[\gamma_1^i(s^i) + \cdots + \gamma_i^i(s^i)\big]\, a_{s^i}^i = m.$$

Since the paths are tight, the sums (equal to the weights of columns):

$$\gamma_1^i(s^i) + \cdots + \gamma_i^i(s^i)$$

jump of a single unit as $s^i = 0, 1, 2, \dots, \tau^i$, and to be precise, they take the following exact values:

$$\gamma_1^i(s^i) + \cdots + \gamma_i^i(s^i) = \begin{cases} \sum_{l=1}^{i} \mu_l^i & \text{for } \ s^i = 0 \\ 1 + \sum_{l=1}^{i} \mu_l^i & \text{for } \ s^i = 1 \\ 2 + \sum_{l=1}^{i} \mu_l^i & \text{for } \ s^i = 2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \kappa + \sum_{l=1}^{i-1} \mu_l^i & \text{for } \ s^i = \tau^i, \end{cases}$$

independently of the paths. In other words:

$$\gamma_1^i(s^i) + \cdots + \gamma_i^i(s^i) = s^i + \mu_1^i + \cdots + \mu_i^i,$$

with of course at the end (as already written):

$$\begin{aligned} \gamma_1^i(\tau^i) + \cdots + \gamma_i^i(\tau^i) &= \kappa + \sum_{l=1}^{i-1} \mu_l^{i-1} - \sum_{l=1}^{i} \mu_l^i + \mu_1^i + \cdots + \mu_i^i \\ &= \kappa + \mu_1^{i-1} + \cdots + \mu_{i-1}^{i-1}. \end{aligned}$$

Recalling that[24]:

$$\big(\ell_i(\mathsf{YT}) - \ell_{i+1}(\mathsf{YT})\big)^{\alpha_i'} = \big(a_0^i + \cdots + a_{\tau^i}^i\big)^{\alpha_i'} \qquad\qquad {\scriptstyle (1 \leqslant i \leqslant n)},$$

we may therefore write:

$$\begin{aligned} &\sum_{\mathsf{YT} \in \mathsf{YT}_{\kappa,m}^{\max}(\mu_l^i)} \big(\ell_1(\mathsf{YT}) - \ell_2(\mathsf{YT})\big)^{\alpha_1'} \cdots \big(\ell_{n-1}(\mathsf{YT}) - \ell_n(\mathsf{YT})\big)^{\alpha_{n-1}'} \big(\ell_n(\mathsf{YT})\big)^{\alpha_n'} = \\ &= \sum_{\gamma^1(s^1)} \cdots \sum_{\gamma^n(s^n)} \sum_{\sum_{i=1}^n \sum_{0 \leqslant s^i \leqslant \tau^i} [s^i + \mu_1^i + \cdots + \mu_i^i] a_{s^i}^i = m} \prod_{i=1}^{n} \big(a_0^i + \cdots + a_{\tau^i}^i\big)^{\alpha_i'}. \end{aligned}$$

[24] By convention, $\ell_{n+1} = \ell_{n+1}(\mathsf{YT}) = 0$ always.

Observing that the last written sum is independent of the paths, each one of the first n sums $\sum_{\gamma^i(s^i)}$ then collapses as just multiplication by the number of considered paths $\gamma^i(s^i)$. Thus, let $N_{\mu_1^1}^{\kappa}$ denote the number[25] of tight paths $\gamma^1(s^1)$ from μ_1^1 to κ; let $N_{\mu_1^2,\mu_2^2}^{\mu_1^1,\kappa}$ denote the number of tight paths $\gamma^2(s^2)$ from the column ${}^t(\mu_1^2,\mu_2^2)$, where ${}^t(\bullet)$ denotes transposition, to the column ${}^t(\mu_1^1,\kappa)$; and generally, let:

$$\boxed{N_{\mu_1^i,\mu_2^i,\dots,\mu_{i-1}^i,\mu_i^i}^{\mu_1^{i-1},\mu_2^{i-1},\dots,\mu_{i-1}^{i-1},\kappa}}$$

denote the number of tight paths $\gamma^i(s^i)$ from the column ${}^t\big(\mu_1^i,\mu_2^i,\dots,\mu_{i-1}^i,\mu_i^i\big)$ to the column ${}^t\big(\mu_1^{i-1},\mu_2^{i-1},\dots,\mu_{i-1}^{i-1},\kappa\big)$, with the natural convention that, for $i=n$, one has the notational equivalence:

$${}^t\big(\mu_1^n,\mu_2^n,\dots,\mu_{n-1}^n,\mu_n^n\big)\equiv{}^t\big(1,2,\dots,n-1,n\big).$$

Then with such notations, we may represent our sum as:

$$\begin{aligned}
&\sum_{\gamma^1(s^1)}\cdots\sum_{\gamma^n(s^n)}\ \sum_{\sum_{i=1}^n\sum_{0\leqslant s^i\leqslant\tau^i}\left[s^i+\mu_1^i+\cdots+\mu_i^i\right]a_{s^i}^i=m}\ \prod_{i=1}^n\big(a_0^i+\cdots+a_{\tau^i}^i\big)^{\alpha_i'}=\\
&=N_{\mu_1^1}^{\kappa}\,N_{\mu_1^2,\mu_2^2}^{\mu_1^1,\kappa}N_{\mu_1^3,\mu_2^3,\mu_3^3}^{\mu_1^2,\mu_2^2,\kappa}\cdots N_{1,\dots,n-1,n}^{\mu_1^{n-1},\dots,\mu_{n-1}^{n-1},\kappa}\cdot\\
&\qquad\cdot\sum_{\sum_{i=1}^n\sum_{0\leqslant s^i\leqslant\tau^i}\left[s^i+\mu_1^i+\cdots+\mu_i^i\right]a_{s^i}^i=m}\ \prod_{i=1}^n\big(a_0^i+\cdots+a_{\tau^i}^i\big)^{\alpha_i'}.
\end{aligned}$$

In conclusion, remembering the dropped $\sum_{\mu_l^i\in\nabla_{n,\kappa}}$, we have established that:

$$\begin{aligned}
&\sum_{\mu_l^i\in\nabla_{n,\kappa}}\ \sum_{\mathsf{YT}\in\mathsf{YT}_{\kappa,m}^{\max}(\mu_l^i)}\big(\ell_1(\mathsf{YT})-\ell_2(\mathsf{YT})\big)^{\alpha_1'}\cdots\big(\ell_{n-1}(\mathsf{YT})-\ell_n(\mathsf{YT})\big)^{\alpha_2'}\big(\ell_n(\mathsf{YT})\big)^{\alpha_n'}=\\
&=\sum_{\mu_l^i\in\nabla_{n,\kappa}}N_{\mu_1^1}^{\kappa}\,N_{\mu_1^2,\mu_2^2}^{\mu_1^1,\kappa}N_{\mu_1^3,\mu_2^3,\mu_3^3}^{\mu_1^2,\mu_2^2,\kappa}\cdots N_{1,\dots,n-1,n}^{\mu_1^{n-1},\dots,\mu_{n-1}^{n-1},\kappa}\cdot\\
&\qquad\cdot\sum_{\sum_{i=1}^n\sum_{0\leqslant s^i\leqslant\tau^i}\left[s^i+\mu_1^i+\cdots+\mu_i^i\right]a_{s^i}^i=m}\ \prod_{i=1}^n\big(a_0^i+\cdots+a_{\tau^i}^i\big)^{\alpha_i'}.
\end{aligned}$$

Approximating sums by integrals. If we now set, similarly as in §3:

$$b_{s^i}^i:=\frac{1}{m}\,a_{s^i}^i,$$

[25] In fact trivially, $N_{\mu_1^1}^{\kappa}=1$.

the sum of the last line of the preceding, boxed equation:

$$\mathsf{S}_{n,\kappa,m}^{\alpha_1',\dots,\alpha_n'}\big(\mu_l^i\big) := \sum_{\sum_{i=1}^n \sum_{0\leqslant s^i\leqslant \tau^i}\left[s^i+\mu_1^i+\cdots+\mu_i^i\right]a_{s^i}^i=m} \prod_{i=1}^n \big(a_0^i+\cdots+a_{\tau^i}^i\big)^{\alpha_i'}$$

becomes:

$$\begin{aligned}\mathsf{S}_{n,\kappa,m}^{\alpha_1',\dots,\alpha_n'}\big(\mu_l^i\big) &= \sum_{\sum_{i=1}^n \sum_{0\leqslant s^i\leqslant \tau^i}\left[s^i+\mu_1^i+\cdots+\mu_i^i\right]\frac{a_{s^i}^i}{m}=1} m^{\alpha_1'+\cdots+\alpha_n'} \prod_{i=1}^n \bigg(\sum_{0\leqslant s^i\leqslant\tau^i}\frac{a_{s^i}^i}{m}\bigg)^{\alpha_i'} \\ &= m^{\frac{n(n+1)}{2}} \sum_{\sum_{i=1}^n \sum_{0\leqslant s^i\leqslant \tau^i}\left[s^i+\mu_1^i+\cdots+\mu_i^i\right]\frac{a_{s^i}^i}{m}=1} \prod_{i=1}^n \bigg(\sum_{0\leqslant s^i\leqslant\tau^i}\frac{a_{s^i}^i}{m}\bigg)^{\alpha_i'}.\end{aligned}$$

Approximating the so obtained sum by a Riemann integral[26], we get up to a negligible power of m:

$$\begin{aligned}\mathsf{S}_{n,\kappa,m}^{\alpha_1',\dots,\alpha_n'}\big(\mu_l^i\big) &= \mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big) + m^{\frac{n(n+1)}{2}}\cdot m^{n\kappa-\frac{n(n-1)}{2}-1}\cdot \\ &\qquad\cdot\int_{\sum_{i=1}^n \sum_{0\leqslant s^i\leqslant \tau^i}\left[s^i+\mu_1^i+\cdots+\mu_i^i\right]b_{s^i}^i=1} \prod_{i=1}^n \bigg(\sum_{0\leqslant s^i\leqslant\tau^i} b_{s^i}^i\bigg)^{\alpha_i'}\end{aligned}$$

Performing next the changes of variables:

$$c_{s^i}^i := \big[s^i+\mu_1^i+\cdots+\mu_i^i\big]b_{s^i}^i,$$

whence:

$$b_{s^i}^i = \frac{c_{s^i}^i}{s^i+\mu_1^i+\cdots+\mu_i^i},$$

we transform the integral as:

$$\begin{aligned}\mathsf{S}_{n,\kappa,m}^{\alpha_1',\dots,\alpha_n'}\big(\mu_l^i\big) &= m^{(\kappa+1)n-1}\cdot \prod_{0\leqslant s^1\leqslant\tau^1}\frac{1}{s^1+\mu_1^1}\cdots\prod_{0\leqslant s^n\leqslant\tau^n}\frac{1}{s^n+\mu_1^n+\cdots+\mu_n^n}\cdot \\ &\qquad\cdot\int_{c_0^1+\cdots+c_{\tau^1}^1+\cdots\cdots+c_0^n+\cdots+c_{\tau^n}^n=1}\prod_{i=1}^n\bigg(\sum_{0\leqslant s^i\leqslant\tau^i}\frac{c_{s^i}^i}{s^i+\mu_1^i+\cdots+\mu_i^i}\bigg)^{\alpha_i'} dc' + \\ &\qquad\qquad + \mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big),\end{aligned}$$

where:

$$dc' := dc_0^1\cdots dc_{\tau^1}^1\cdots\cdots dc_0^n\cdots dc_{\tau^n}^n.$$

[26] A different view may be found in [2].

Using the multinomial formula, we now expand the product of all the α_i'-th powers under the sum in the second line above:

$$\prod_{i=1}^{n}\bigg[\sum_{0\leqslant s^i\leqslant\tau^i}\frac{c_{s^i}^i}{s^i+\mu_1^i+\cdots+\mu_i^i}\bigg]^{\alpha_i'}=$$

$$=\prod_{i=1}^{n}\bigg[\sum_{q_0^i+\cdots+q_{\tau^i}^i=\alpha_i'}\frac{\alpha_i'!}{q_0^i!\cdots q_{\tau^i}^i!}\prod_{0\leqslant s^i\leqslant\tau^i}\Big(\frac{c_{s^i}^i}{s^i+\mu_1^i+\cdots+\mu_i^i}\Big)^{q_{s^i}^i}\bigg]$$

$$=\sum_{q_0^1+\cdots+q_{\tau^1}^1=\alpha_1'}\cdots\sum_{q_0^n+\cdots+q_{\tau^n}^n=\alpha_n'}\frac{\alpha_1'!}{q_0^1!\cdots q_{\tau^1}^1!}\cdots\frac{\alpha_n'!}{q_0^n!\cdots q_{\tau^n}^n!}\cdot$$

$$\cdot\prod_{0\leqslant s^1\leqslant\tau^1}\Big(\frac{c_{s^1}^1}{s^1+\mu_1^1}\Big)^{q_{s^1}^1}\cdots\prod_{0\leqslant s^n\leqslant\tau^n}\Big(\frac{c_{s^n}^n}{s^n+\mu_1^n+\cdots+\mu_n^n}\Big)^{q_{s^n}^n}$$

$$=\sum_{q_0^1+\cdots+q_{\tau^1}^1=\alpha_1'}\cdots\sum_{q_0^n+\cdots+q_{\tau^n}^n=\alpha_n'}\frac{\alpha_1'!}{q_0^1!\cdots q_{\tau^1}^1!}\cdots\frac{\alpha_n'!}{q_0^n!\cdots q_{\tau^n}^n!}\cdot$$

$$\cdot\prod_{0\leqslant s^1\leqslant\tau^1}\frac{1}{\big(s^1+\mu_1^1\big)^{q_{s^1}^1}}\cdots\prod_{0\leqslant s^n\leqslant\tau^n}\frac{1}{\big(s^n+\mu_1^n+\cdots+\mu_n^n\big)^{q_{s^n}^n}}\cdot$$

$$\cdot\prod_{0\leqslant s^1\leqslant\tau^1}\big(c_{s^1}^1\big)^{q_{s^1}^1}\cdots\prod_{0\leqslant s^n\leqslant\tau^n}\big(c_{s^n}^n\big)^{q_{s^n}^n}.$$

After these expansions are done, in order to complete the computation of $\mathsf{S}_{n,\kappa,m}^{\alpha_1',\dots,\alpha_n'}\big(\mu_l^i\big)$, we are left with the task of computing the integrals:

$$\int_{c_0^1+\cdots+c_{\tau^1}^1+\cdots\cdots+c_0^n+\cdots+c_{\tau^n}^n=1}\prod_{0\leqslant s^1\leqslant\tau^1}\big(c_{s^1}^1\big)^{q_{s^1}^1}\cdots\prod_{0\leqslant s^n\leqslant\tau^n}\big(c_{s^n}^n\big)^{q_{s^n}^n}\,dc'.$$

To this aim, we simply apply the elementary lemma on page 18, and this then yields to us the desired value:

$$\int_{c_0^1+\cdots+c_{\tau^1}^1+\cdots\cdots+c_0^n+\cdots+c_{\tau^n}^n=1}\prod_{0\leqslant s^1\leqslant\tau^1}\big(c_{s^1}^1\big)^{q_{s^1}^1}\cdots\prod_{0\leqslant s^n\leqslant\tau^n}\big(c_{s^n}^n\big)^{q_{s^n}^n}=$$

$$=\frac{q_0^1!\,\cdots\,q_{\tau^1}^1!\,\cdots\cdots\,q_0^n!\,\cdots\,q_{\tau^n}^n!}{(q_0^1+\cdots+q_{\tau^1}^1+\cdots\cdots+q_0^n+\cdots+q_{\tau^n}^n+(1+\tau^1)+\cdots+(1+\tau^n)-1)!}$$

$$=\frac{q_0^1!\,\cdots\,q_{\tau^1}^1!\,\cdots\cdots\,q_0^n!\,\cdots\,q_{\tau^n}^n!}{(\alpha_1'+\cdots\cdots+\alpha_n'+n\kappa-\frac{n(n-1)}{2}-1)!}$$

$$=\frac{q_0^1!\,\cdots\,q_{\tau^1}^1!\,\cdots\cdots\,q_0^n!\,\cdots\,q_{\tau^n}^n!}{\big((\kappa+1)n-1\big)!},$$

since $q_0^i+\cdots+q_{\tau^i}^i=\alpha_i'$ and since $\alpha_1'+\cdots+\alpha_n'=\frac{n(n+1)}{2}$.

Resuming what has been done, we therefore get:

$$
\begin{aligned}
\mathsf{S}_{n,\kappa,m}^{\alpha_1',\dots,\alpha_n'}\big(\mu_i^l\big) = m^{(\kappa+1)n-1}\cdot \prod_{0\leqslant s^1\leqslant\tau^1}\frac{1}{s^1+\mu_1^1}\cdots\prod_{0\leqslant s^n\leqslant\tau^n}\frac{1}{s^n+\mu_1^n+\cdots+\mu_n^n}\cdot \\
\cdot\sum_{q_0^1+\cdots+q_{\tau^1}^1=\alpha_1'}\cdots\sum_{q_0^n+\cdots+q_{\tau^n}^n=\alpha_n'}\frac{\alpha_1'!}{\underline{q_0^1!\cdots q_{\tau^1}^1!}_\circ}\cdots\frac{\alpha_n'!}{\underline{q_0^n!\cdots q_{\tau^n}^n!}_\circ}\cdot \\
\cdot\prod_{0\leqslant s^1\leqslant\tau^1}\frac{1}{\big(s^1+\mu_1^1\big)^{q_{s^1}^1}}\cdots\prod_{0\leqslant s^n\leqslant\tau^n}\frac{1}{\big(s^n+\mu_1^n+\cdots+\mu_n^n\big)^{q_{s^n}^n}}\cdot \\
\cdot\frac{\underline{q_0^1!\,\cdots\,q_{\tau^1}^1!}_\circ\;\cdots\cdots\;\underline{q_0^n!\,\cdots\,q_{\tau^n}^n!}_\circ}{\big((\kappa+1)n-1\big)!}+ \\
+\mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big).
\end{aligned}
$$

The products of the factorials of the $q_{s^i}^i$ disappear and a reorganization gives:

$$
\begin{aligned}
\mathsf{S}_{n,\kappa,m}^{\alpha_1',\dots,\alpha_n'}\big(\mu_i^l\big) = \frac{m^{(\kappa+1)n-1}}{\big((\kappa+1)n-1\big)!}\cdot \\
\cdot\prod_{0\leqslant s^1\leqslant\tau^1}\frac{1}{s^1+\mu_1^1}\cdots\prod_{0\leqslant s^n\leqslant\tau^n}\frac{1}{s^n+\mu_1^n+\cdots+\mu_n^n}\cdot \\
\cdot\,\alpha_1'!\cdots\alpha_n'!\cdot\sum_{q_0^1+\cdots+q_{\tau^1}^1=\alpha_1'}\cdots\sum_{q_0^n+\cdots+q_{\tau^n}^n=\alpha_n'}\Bigg(\\
\Bigg(\prod_{0\leqslant s^1\leqslant\tau^1}\frac{1}{\big(s^1+\mu_1^1\big)^{q_{s^1}^1}}\cdots\prod_{0\leqslant s^n\leqslant\tau^n}\frac{1}{\big(s^n+\mu_1^n+\cdots+\mu_n^n\big)^{q_{s^n}^n}}\Bigg)+ \\
+\mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big).
\end{aligned}
$$

Symbolically, instead of:

$$
\prod_{0\leqslant s^1\leqslant\tau^1}\frac{1}{s^1+\mu_1^1}\prod_{0\leqslant s^2\leqslant\tau^2}\frac{1}{s^2+\mu_1^2+\mu_2^2}\cdots\prod_{0\leqslant s^n\leqslant\tau^n}\frac{1}{s^n+\mu_1^n+\cdots+\mu_n^n},
$$

we shall write without any risk of ambiguity:

$$
\frac{1}{\kappa\cdots\mu_1^1}\,\frac{1}{(\kappa+\mu_1^1)\cdots(\mu_2^2+\mu_1^2)}\cdots\frac{1}{(\kappa+\mu_{n-1}^{n-1}+\cdots+\mu_1^{n-1})\cdots(n+(n-1)+\cdots+1)},
$$

the dots in the denominators meaning that one takes the product of all integers, decreasingly, that are extant between the two written extremal integers. In conclusion, we have established that:

$$
\boxed{
\begin{aligned}
&\sum_{\substack{\mathsf{YT\ semi-standard}\\ \mathsf{weight(YT)}=m}} \big(\ell_1(\mathsf{YT})-\ell_2(\mathsf{YT})\big)^{\alpha_1'}\cdots\big(\ell_{n-1}(\mathsf{YT})-\ell_n(\mathsf{YT})\big)^{\alpha_{n-1}'}\big(\ell_n(\mathsf{YT})\big)^{\alpha_n'} = \\
&= \frac{m^{(\kappa+1)n-1}}{\big((\kappa+1)n-1\big)!}\sum_{\mu_l^i\in\nabla_{n,\kappa}} \frac{N_{\mu_1^1}^{\kappa}}{\kappa\cdots\mu_1^1}\,\frac{N_{\mu_1^2,\mu_2^2}^{\mu_1^1,\kappa}}{(\kappa+\mu_1^1)\cdots(\mu_2^2+\mu_1^2)}\cdots \\
&\qquad\cdots\frac{N_{\mu_1^n,\dots,\mu_{n-1}^n,\mu_n^n}^{\mu_1^{n-1},\dots,\mu_{n-1}^{n-1},\kappa}}{(\kappa+\mu_{n-1}^{n-1}+\cdots+\mu_1^{n-1})\cdots(\mu_n^n+\mu_{n-1}^n+\cdots+\mu_1^n)}\cdot \\
&\alpha_1'!\cdots\alpha_n'!\cdot\sum_{q_0^1+\cdots+q_{\tau^1}^1=\alpha_1'}\cdots\sum_{q_0^n+\cdots+q_{\tau^n}^n=\alpha_n'} \\
&\bigg(\prod_{0\leqslant s^1\leqslant\tau^1}\frac{1}{\big(s^1+\mu_1^1\big)^{q_{s^1}^1}}\cdots\prod_{0\leqslant s^n\leqslant\tau^n}\frac{1}{\big(s^n+\mu_1^n+\cdots+\mu_{n-1}^n+\mu_n^n\big)^{q_{s^n}^n}}\bigg)+ \\
&\qquad\qquad\qquad\qquad + \mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big),
\end{aligned}
}
\tag{24}
$$

where we recall for completeness that $\tau^i = \kappa + \sum_{l=1}^{i-1}\mu_l^{i-1} - \sum_{l=1}^{i}\mu_l^i$ for convenient abbreviation.

§9. Number of tight paths in semi-standard Young tableaux

Summary. Thus, we are left with the task of computing or of majorating, for any $\alpha_1',\dots,\alpha_n'$ with $\alpha_1'+\cdots+\alpha_n' = \frac{n(n+1)}{2}$, sums:

$$
\begin{aligned}
\square_{n,\kappa}^{\alpha_1',\dots,\alpha_n'} := \sum_{\mu_l^i\in\nabla_{n,\kappa}} (\kappa!)^n\cdot\frac{N_{\mu_1^1}^{\kappa}}{\kappa\cdots\mu_1^1}\,\frac{N_{\mu_1^2,\mu_2^2}^{\mu_1^1,\kappa}}{(\kappa+\mu_1^1)\cdots(\mu_2^2+\mu_1^2)}\cdots \\
\cdots\frac{N_{\mu_1^n,\dots,\mu_{n-1}^n,\mu_n^n}^{\mu_1^{n-1},\dots,\mu_{n-1}^{n-1},\kappa}}{(\kappa+\mu_{n-1}^{n-1}+\cdots+\mu_1^{n-1})\cdots(\mu_n^n+\mu_{n-1}^n+\cdots+\mu_1^n)}\cdot \\
\alpha_1'!\cdots\alpha_n'!\cdot\sum_{q_0^1+\cdots+q_{\tau^1}^1=\alpha_1'}\cdots\sum_{q_0^n+\cdots+q_{\tau^n}^n=\alpha_n'} \\
\bigg(\prod_{0\leqslant s^1\leqslant\tau^1}\frac{1}{\big(s^1+\mu_1^1\big)^{q_{s^1}^1}}\cdots\prod_{0\leqslant s^n\leqslant\tau^n}\frac{1}{\big(s^n+\mu_1^n+\cdots+\mu_{n-1}^n+\mu_n^n\big)^{q_{s^n}^n}}\bigg)
\end{aligned}
$$

in which the weight m has completely disappeared, while only the dimension n and the jet order κ remain present.

At first, we would like to remind from the two theorems on p. 31 and on p. 37 that the basic numerical sums $\sum M\cdot\ell^{\alpha}$ we must compute were in fact born when $\alpha_1+\cdots+\alpha_n$ is maximal equal to $\frac{n(n+1)}{2}$, after rewriting in terms of

$\ell_1 - \ell_2, \ldots, \ell_{n-1} - \ell_n$ and ℓ_n expressions of the form:

$$\Big[\prod_{1\leqslant i<j\leqslant n} (\ell_i - \ell_j)\Big] \cdot \ell_1^{\beta_1} \cdots \ell_n^{\beta_n}$$

that are multiple of the product of the $\ell_i - \ell_j$ with $\beta_1 + \cdots + \beta_n = n$. Each $\ell_i - \ell_j$ then writes as $\ell_i - \ell_{i+1} + \cdots + \ell_{j-1} - \ell_j$ with no ℓ_n at all, and it follows that after the rewriting, the exponent α_n' of ℓ_n is at most equal to n:

(25)

$$\begin{aligned}\Big[\prod_{1\leqslant i<j\leqslant n} (\ell_i - \ell_j)\Big] &\cdot \ell_1^{\beta_1} \cdots \ell_{n-1}^{\beta_{n-1}} \ell_n^{\beta_n} \\ &\leqslant \mathsf{Constant}_n \cdot \sum_{\substack{\alpha_1' + \cdots + \alpha_{n-1}' + \alpha_n' = \frac{n(n+1)}{2} \\ \alpha_n' \leqslant n}} (\ell_1 - \ell_2)^{\alpha_1'} \cdots (\ell_{n-1} - \ell_n)^{\alpha_{n-1}'} \ell_n^{\alpha_n'}.\end{aligned}$$

Thus, the sums $\square_{n,\kappa}^{\alpha_1',\ldots,\alpha_n'}$ we consider are such that $\alpha_1' + \cdots + \alpha_n' = \frac{n(n+1)}{2}$ and $\alpha_n' \leqslant n$.

Logarithmic equivalents. Next, we observe that for any integer $\alpha' \geqslant 1$, as soon as $\tau \geqslant \alpha'$, one has:

$$\sum_{q_0+\cdots+q_\tau=\alpha'} \frac{1}{(k)^{q_0}\cdots(k+\tau)^{q_\tau}} = \frac{1}{\alpha'!}\big[\log(k+\tau) - \log(k)\big]^{\alpha'} + \mathsf{O}_{\alpha'}\Big(\big[\log(k+\tau) - \log(k)\big]^{\alpha'-1}\Big).$$

If $\tau \leqslant \alpha' - 1$, this sum is smaller than the written power of a difference between two logarithms. Since our goal now will be to establish only an inequality of the form:

$$\boxed{\square_{n,\kappa}^{\alpha_1',\ldots,\alpha_n'} \leqslant \mathsf{Constant}_n \cdot (\log\kappa)^{\alpha_n'}}, \tag{26}$$

in which no particular knowledge about the $\mathsf{Constant}_n$ will be required, again with $\alpha_1' + \cdots + \alpha_n' = \frac{n(n+1)}{2}$ and with $\alpha_n' \leqslant n$, it will even suffice to observe that the last two lines in the definition of $\square_{n,\kappa}^{\alpha_1',\ldots,\alpha_n'}$ enjoy a majoration of the sort:

$$\begin{aligned}&\alpha_1'!\,\alpha_2'!\cdots\alpha_n'! \cdot \sum_{q_0^1+\cdots+q_{\tau^1}^1=\alpha_1'}\ \sum_{q_0^2+\cdots+q_{\tau^2}^2=\alpha_2'} \cdots \sum_{q_0^n+\cdots+q_{\tau^n}^n=\alpha_n'}\ \prod_{0\leqslant s^1\leqslant\tau^1} \frac{1}{(s^1+\mu_1^1)^{q_{s^1}^1}} \\ &\prod_{0\leqslant s^2\leqslant\tau^2} \frac{1}{(s^2+\mu_1^2+\mu_2^2)^{q_{s^2}^2}} \cdots \prod_{0\leqslant s^n\leqslant\tau^n} \frac{1}{(s^n+\mu_1^n+\cdots+\mu_{n-1}^n+\mu_n^n)^{q_{s^n}^n}} \leqslant \\ &\leqslant \mathsf{Constant}_n \cdot \big[\log(\kappa) - \log(\mu_1^1)\big]^{\alpha_1'}\big[\log(\kappa+\mu_1^1) - \log(\mu_2^2+\mu_1^2)\big]^{\alpha_2'}\cdots \\ &\quad\cdots\big[\log(\kappa+\mu_{n-1}^{n-1}+\cdots+\mu_1^{n-1}) - \log(n+\cdots+2+1)\big]^{\alpha_n'}.\end{aligned}$$

Consequently, we are left with establishing the following proposition.

Proposition. *Let $\alpha_1', \ldots, \alpha_n' \in \mathbb{N}$ with $\alpha_1' + \cdots + \alpha_n' = \frac{n(n+1)}{2}$ and with $\alpha_n' \leqslant n$. Then for $\kappa \geqslant n$, one has the majoration:*

$$
\begin{aligned}
\widetilde{\square}_{n,\kappa}^{\alpha_1',\dots,\alpha_n'} := \sum_{\mu_l^i\in\nabla_{n,\kappa}} & (\kappa!)^n \cdot \frac{N_{\mu_1^1}^{\kappa}}{\kappa\cdots\mu_1^1}\,\frac{N_{\mu_1^2,\mu_2^2}^{\mu_1^1,\kappa}}{(\kappa+\mu_1^1)\cdots(\mu_2^2+\mu_1^2)}\cdots \\
& \cdots\frac{N_{\mu_1^n,\dots,\mu_{n-1}^n,\mu_n^n}^{\mu_1^{n-1},\dots,\mu_{n-1}^{n-1},\kappa}}{(\kappa+\mu_{n-1}^{n-1}+\cdots+\mu_1^{n-1})\cdots(\mu_n^n+\mu_{n-1}^n+\cdots+\mu_1^n)}\cdot \\
& \big[\log(\kappa)-\log(\mu_1^1)\big]^{\alpha_1'}\big[\log(\kappa+\mu_1^1)-\log(\mu_2^2+\mu_1^2)\big]^{\alpha_2'}\cdots \\
& \cdots\big[\log(\kappa+\mu_{n-1}^{n-1}+\cdots+\mu_1^{n-1})-\log(n+\cdots+2+1)\big]^{\alpha_n'}\leqslant \\
\leqslant \mathsf{Constant}_n\cdot(\log\kappa)^{\alpha_n'}.
\end{aligned}
$$

Tight paths. According to the proposition on p. 55 and to the definition made on p. 59, the integer $N_{\mu_1^1}^{\kappa}$ denotes the number of tight paths from the column μ_1^1 to the column κ, hence it is equal to 1. When the dimension n is equal to 2, one may show that:

$$
N_{1,2}^{\mu_1^1,\kappa} = \tfrac{(\kappa+\mu_1^1)!}{(\kappa-3)!\,(\mu_1^1-1)!} - \tfrac{(\kappa+\mu_1^1-4)!}{(\kappa-1)!\,(\mu_1^1-3)!}.
$$

In higher dimensions, the exact computation of the numbers $N_{\mu_1^2,\mu_2^2}^{\mu_1^1,\kappa}$, $N_{\mu_1^3,\mu_2^3,\mu_3^3}^{\mu_1^2,\mu_2^2,\kappa}$, ... may certainly be done and it involves only differences of multinomial coefficients, but very many cases are to be considered according to certain inequalities between the μ_i^j. However, after some explorations, it appears that in order to get the majoration claimed by the proposition, it suffices to *majorate* these numbers uniformly as follows.

Majoration of the tight path numbers $N_{\mu_1^i,\dots,\mu_{i-1}^i,\mu_i^i}^{\mu_1^{i-1},\dots,\mu_{i-1}^{i-1},\kappa}$. By definition, $N_{\mu_1^i,\dots,\mu_{i-1}^i,\mu_i^i}^{\mu_1^{i-1},\dots,\mu_{i-1}^{i-1},\kappa}$ counts the number of strictly increasing tight paths from the column $\big[\mu_1^i\ \cdots\ \mu_{i-1}^i\ \mu_i^i\big]^{\text{transposed}}$ to the column $\big[\mu_1^{i-1}\ \cdots\ \mu_{i-1}^{i-1}\ \kappa\big]^{\text{transposed}}$ in the i-dimensional lattice $\mathbb{N}^i$ with the supplementary constraint that at each point $\big[\gamma_1^i(s^i)\ \cdots\ \gamma_{i-1}^i(s^i)\ \gamma_i(s^i)\big]^{\text{transposed}}$ of the path, the inequalities $\gamma_1^i(s^i)<\cdots<\gamma_{i-1}^i(s^i)<\gamma_i(s^i)$ (strict increase inside columns, downward) must be satisfied.

If we relax this last constraint, there are clearly more paths. But the number of strictly increasing paths in a complete lattice is elementarily computed. Thus we deduce that:

$$
N_{\mu_1^i,\dots,\mu_{i-1}^i,\mu_i^i}^{\mu_1^{i-1},\dots,\mu_{i-1}^{i-1},\kappa} \leqslant \tfrac{(\mu_1^{i-1}-\mu_1^i+\cdots+\mu_{i-1}^{i-1}-\mu_{i-1}^i+\kappa-\mu_i^i)!}{(\mu_1^{i-1}-\mu_1^i)!\cdots(\mu_{i-1}^{i-1}-\mu_{i-1}^i)!\,(\kappa-\mu_i^i)!}.
$$

Removal of α_n'. On the other hand, for any choice of $\mu_{n-1}^{n-1},\dots,\mu_1^{n-1}$ as in the proposition on p. 55, the last difference between logarithms:

$$
\big[\log(\kappa+\mu_{n-1}^{n-1}+\cdots+\mu_1^{n-1})-\log(n+\cdots+2+1)\big]^{\alpha_n'}
$$

enjoys, when $\kappa \gg n$, a doubly controlling inequality of the form:

$$\frac{1}{\mathsf{C}_n}\cdot[\log(\kappa)]^{\alpha_n'} \leqslant \big[\log(\kappa+\mu_{n-1}^{n-1}+\cdots+\mu_1^{n-1})-\log(n+\cdots+2+1)\big]^{\alpha_n'} \leqslant \mathsf{C}_n\cdot[\log(\kappa)]^{\alpha_n'}$$

where the constant $\mathsf{C}_n > 1$ can be chosen arbitrarily close to 1 provided that $\kappa \geqslant \kappa_{\mathsf{C}_n} \gg n$ is large enough. Consequently, in order to establish the inequality boxed on p. 64, it suffices now to establish the following concrete proposition, in which $\alpha_n' = 0$.

Proposition. *Let $\alpha_1',\dots,\alpha_{n-1}' \in \mathbb{N}$ with $\alpha_1'+\cdots+\alpha_{n-1}' = \frac{n(n+1)}{2}$ and assume $\kappa \geqslant n$. Then the following sum is bounded independently of κ:*

$$\begin{aligned}
\Delta_{n,\kappa}^{\alpha_1',\dots,\alpha_{n-1}',0} := \sum_{\mu_l^i\in\nabla_{n,\kappa}} (\kappa!)^n\cdot 1\,\frac{(\mu_1^1-1)!}{\kappa!}\cdot\frac{(\mu_1^1-\mu_1^2+\kappa-\mu_2^2)!}{(\mu_1^1-\mu_1^2)!\,(\kappa-\mu_2^2)!}\,\frac{(\mu_2^2+\mu_1^2-1)!}{(\kappa+\mu_1^1)!}\,\cdot \\
\cdot\,\frac{(\mu_1^2-\mu_1^3+\mu_2^2-\mu_2^3+\kappa-\mu_3^3)!}{(\mu_1^2-\mu_1^3)!\,(\mu_2^2-\mu_2^3)!\,(\kappa-\mu_3^3)!}\,\frac{(\mu_3^3+\mu_2^3+\mu_1^3-1)!}{(\kappa+\mu_2^2+\mu_1^2)!}\cdots \\
\cdots\frac{(\mu_1^{n-2}-\mu_1^{n-1}+\cdots+\mu_{n-2}^{n-2}-\mu_{n-2}^{n-1}+\kappa-\mu_{n-1}^{n-1})!}{(\mu_1^{n-2}-\mu_1^{n-1})!\cdots(\mu_{n-2}^{n-2}-\mu_{n-2}^{n-1})!\,(\kappa-\mu_{n-1}^{n-1})!}\,\frac{(\mu_{n-1}^{n-1}+\mu_{n-2}^{n-1}+\cdots+\mu_1^{n-1}-1)!}{(\kappa+\mu_{n-2}^{n-2}+\cdots+\mu_1^{n-2})!}\,\cdot \\
\cdot\,\frac{(\mu_1^{n-1}-\mu_1^n+\cdots+\mu_{n-1}^{n-1}-\mu_{n-1}^n+\kappa-\mu_n^n)!}{(\mu_1^{n-1}-\mu_1^n)!\cdots(\mu_{n-1}^{n-1}-\mu_{n-1}^n)!\,(\kappa-\mu_n^n)!}\,\frac{(\mu_n^n+\mu_{n-1}^n+\cdots+\mu_1^n-1)!}{(\kappa+\mu_{n-1}^{n-1}+\cdots+\mu_1^{n-1})!}\,\cdot \\
\cdot\big[\log(\kappa)-\log(\mu_1^1)\big]^{\alpha_1'}\cdot\big[\log(\kappa+\mu_1^1)-\log(\mu_2^2+\mu_1^2)\big]^{\alpha_2'}\cdot \\
\cdot\big[\log(\kappa+\mu_2^2+\mu_1^2)-\log(\mu_3^3+\mu_2^3+\mu_1^3)\big]^{\alpha_3'}\cdots \\
\cdots\big[\log(\kappa+\mu_{n-2}^{n-2}+\cdots+\mu_1^{n-2})-\log(\mu_{n-1}^{n-1}+\mu_{n-2}^{n-1}+\cdots+\mu_1^{n-1})\big]^{\alpha_{n-1}'} \leqslant \\
\leqslant \mathsf{Constant}_n.
\end{aligned}$$

§10. Bounded behavior of plurilogarithmic sums

Simplifying the kernel. To begin with, disregarding the logarithmic factors, or equivalently, considering that $\alpha_1'=\alpha_2'=\alpha_3'=\cdots=\alpha_{n-1}'=0$, we observe that the rational factor simplifies a bit (a factorial $\kappa!$ disappears) and can be majorated as follows:

$$\begin{aligned}
\underline{\kappa!}_{\circ}\,(\kappa!)^{n-2}\,\underline{\kappa!}_{ⓐ}\cdot 1\,\frac{(\mu_1^1-1)!}{\underline{\kappa!}_{\circ}}\cdot\frac{(\mu_1^1-\mu_1^2+\kappa-\mu_2^2)!}{(\mu_1^1-\mu_1^2)!\,(\kappa-\mu_2^2)!}\,\frac{(\mu_2^2+\mu_1^2-1)!}{(\kappa+\mu_1^1)!}\cdots \\
\cdot\,\frac{(\mu_1^2-\mu_1^3+\mu_2^2-\mu_2^3+\kappa-\mu_3^3)!}{(\mu_1^2-\mu_1^3)!\,(\mu_2^2-\mu_2^3)!\,(\kappa-\mu_3^3)!}\,\frac{(\mu_3^3+\mu_2^3+\mu_1^3-1)!}{(\kappa+\mu_2^2+\mu_1^2)!}\cdots \\
\cdots\frac{(\mu_1^{n-2}-\mu_1^{n-1}+\cdots+\mu_{n-2}^{n-2}-\mu_{n-2}^{n-1}+\kappa-\mu_{n-1}^{n-1})!}{(\mu_1^{n-2}-\mu_1^{n-1})!\cdots(\mu_{n-2}^{n-2}-\mu_{n-2}^{n-1})!\,(\kappa-\mu_{n-1}^{n-1})!}\,\frac{(\mu_{n-1}^{n-1}+\mu_{n-2}^{n-1}+\cdots+\mu_1^{n-1}-1)!}{(\kappa+\mu_{n-2}^{n-2}+\cdots+\mu_1^{n-2})!}\,\cdot \\
\cdot\,\frac{\underline{(\mu_1^{n-1}-1+\cdots+\mu_{n-1}^{n-1}-(n-1)+\kappa-n)!}_{ⓒ}}{(\mu_1^{n-1}-1)!\cdots(\mu_{n-1}^{n-1}-(n-1))!\,\underline{(\kappa-n)!}_{ⓐ}}\,\frac{\underline{(n+(n-1)+\cdots+1-1)!}_{ⓑ}}{\underline{(\kappa+\mu_{n-1}^{n-1}+\cdots+\mu_1^{n-1})!}_{ⓒ}} \leqslant \\
\leqslant \mathsf{Constant}_n\cdot(\kappa!)^{n-2}\cdot(\mu_1^1-1)!\cdot\frac{(\mu_1^1-\mu_1^2+\kappa-\mu_2^2)!}{(\mu_1^1-\mu_1^2)!\,(\kappa-\mu_2^2)!}\,\frac{(\mu_2^2+\mu_1^2-1)!}{(\kappa+\mu_1^1)!}\cdots \\
\cdot\,\frac{(\mu_1^2-\mu_1^3+\mu_2^2-\mu_2^3+\kappa-\mu_3^3)!}{(\mu_1^2-\mu_1^3)!\,(\mu_2^2-\mu_2^3)!\,(\kappa-\mu_3^3)!}\,\frac{(\mu_3^3+\mu_2^3+\mu_1^3-1)!}{(\kappa+\mu_2^2+\mu_1^2)!}\cdots
\end{aligned}$$

$$
\begin{aligned}
&\cdots \frac{(\mu_1^{n-2}-\mu_1^{n-1}+\cdots+\mu_{n-2}^{n-2}-\mu_{n-2}^{n-1}+\kappa-\mu_{n-1}^{n-1})!}{(\mu_1^{n-2}-\mu_1^{n-1})!\cdots(\mu_{n-2}^{n-2}-\mu_{n-2}^{n-1})!\,(\kappa-\mu_{n-1}^{n-1})!}\,\frac{(\mu_{n-1}^{n-1}+\mu_{n-2}^{n-1}+\cdots+\mu_1^{n-1}-1)!}{(\kappa+\mu_{n-2}^{n-2}+\cdots+\mu_1^{n-2})!}\cdot \\
&\cdot \frac{1}{(\mu_1^{n-1}-1)!\cdots(\mu_{n-1}^{n-1}-(n-1))!}\cdot \\
&\cdot \frac{\kappa(\kappa-1)\cdots(\kappa-n+1)}{(\kappa+\mu_{n-1}^{n-1}+\cdots+\mu_1^{n-1})\cdots(\kappa+\mu_{n-1}^{n-1}+\cdots+\mu_1^{n-1}-\frac{n(n+1)}{2}+1)},
\end{aligned}
$$

since the two pairs of terms underlined with ⓐ and ⓒ appended can be put at the end and simplified, while the pair of terms with ⓑ appended, equal to the factorial $(\frac{n(n+1)}{2}-1)!$, may be considered as just a $\mathsf{Constant}_n$. But now, the last line is controlled as follows:

$$
\mathsf{C}_n^{-1}\,\kappa^{-\frac{n(n-1)}{2}} \leqslant \frac{\kappa(\kappa-1)\cdots(\kappa-n+1)}{(\kappa+\mu_{n-1}^{n-1}+\cdots+\mu_1^{n-1})\cdots(\kappa+\mu_{n-1}^{n-1}+\cdots+\mu_1^{n-1}-\frac{n(n+1)}{2}+1)} \leqslant \mathsf{C}_n\,\kappa^{-\frac{n(n-1)}{2}},
$$

for some constant $\mathsf{C}_n > 1$. Consequently, we are reduced to the following proposition.

Proposition. *Let $\alpha_1', \dots, \alpha_{n-1}' \in \mathbb{N}$ with $\alpha_1' + \cdots + \alpha_{n-1}' = \frac{n(n+1)}{2}$ and assume $\kappa \geqslant n$. Then the following sum is bounded independently of κ:*

$$
\begin{aligned}
\mathsf{K}^n_{\alpha_1',\dots,\alpha_{n-1}'}(\kappa) := \sum_{\mu_l^i \in \nabla_{n,\kappa}} &\frac{1}{\kappa^{\frac{n(n-1)}{2}}}\cdot(\kappa!)^{n-2}\cdot(\mu_1^1-1)!\cdot \\
&\cdot \frac{(\mu_1^1-\mu_1^2+\kappa-\mu_2^2)!}{(\mu_1^1-\mu_1^2)!\,(\kappa-\mu_2^2)!}\,\frac{(\mu_2^2+\mu_1^2-1)!}{(\kappa+\mu_1^1)!}\cdot\frac{(\mu_1^2-\mu_1^3+\mu_2^2-\mu_2^3+\kappa-\mu_3^3)!}{(\mu_1^2-\mu_1^3)!\,(\mu_2^2-\mu_2^3)!\,(\kappa-\mu_3^3)!}\,\frac{(\mu_3^3+\mu_2^3+\mu_1^3-1)!}{(\kappa+\mu_2^2+\mu_1^2)!}\cdots \\
&\cdots\frac{(\mu_1^{n-2}-\mu_1^{n-1}+\cdots+\mu_{n-2}^{n-2}-\mu_{n-2}^{n-1}+\kappa-\mu_{n-1}^{n-1})!}{(\mu_1^{n-2}-\mu_1^{n-1})!\cdots(\mu_{n-2}^{n-2}-\mu_{n-2}^{n-1})!\,(\kappa-\mu_{n-1}^{n-1})!}\,\frac{(\mu_{n-1}^{n-1}+\mu_{n-2}^{n-1}+\cdots+\mu_1^{n-1}-1)!}{(\kappa+\mu_{n-2}^{n-2}+\cdots+\mu_1^{n-2})!}\cdot \\
&\cdot\frac{1}{(\mu_1^{n-1}-1)!\cdots(\mu_{n-1}^{n-1}-(n-1))!}\cdot \\
&\cdot\big[\log(\kappa)-\log(\mu_1^1)\big]^{\alpha_1'}\cdot\big[\log(\kappa+\mu_1^1)-\log(\mu_2^2+\mu_1^2)\big]^{\alpha_2'}\cdot \\
&\cdot\big[\log(\kappa+\mu_2^2+\mu_1^2)-\log(\mu_3^3+\mu_2^3+\mu_1^3)\big]^{\alpha_3'}\cdots \\
&\cdots\big[\log(\kappa+\mu_{n-2}^{n-2}+\cdots+\mu_1^{n-2})-\log(\mu_{n-1}^{n-1}+\mu_{n-2}^{n-1}+\cdots+\mu_1^{n-1})\big]^{\alpha_{n-1}'} \leqslant \\
&\leqslant \mathsf{Constant}_n.
\end{aligned}
$$

Here, the summation $\sum_{\mu_l^i\in\nabla_{n,\kappa}}$ holds for μ_l^i satisfying the two collections of inequalities (21) and (22), and we may expand it symbolically using two symbols Σ in order to notify well these two conditions:

$$
\sum_{\mu_l^i\in\nabla_{n,\kappa}} \equiv \sum_{\substack{1\leqslant\mu_1^1<\kappa \\ 1\leqslant\mu_1^2<\mu_2^2<\kappa \\ 1\leqslant\mu_1^3<\mu_2^3<\mu_3^3<\kappa \\ \cdots\cdots\cdots\cdots \\ 1\leqslant\mu_1^{n-2}<\mu_2^{n-2}<\mu_3^{n-2}<\cdots<\mu_{n-2}^{n-2}<\kappa \\ 1\leqslant\mu_1^{n-1}<\mu_2^{n-1}<\mu_3^{n-1}<\cdots<\mu_{n-2}^{n-1}<\mu_{n-1}^{n-1}<\kappa}} \quad \sum_{\substack{\mu_1^1\geqslant\mu_1^2\geqslant\mu_1^3\geqslant\cdots\geqslant\mu_1^{n-2}\geqslant\mu_1^{n-1} \\ \mu_2^2\geqslant\mu_2^3\geqslant\cdots\geqslant\mu_2^{n-2}\geqslant\mu_2^{n-1} \\ \mu_3^3\geqslant\cdots\geqslant\mu_3^{n-2}\geqslant\mu_3^{n-1} \\ \cdots\cdots\cdots \\ \mu_{n-2}^{n-2}\geqslant\mu_{n-2}^{n-1}}}
$$

In dimension $n = 2$, the sum writes:

$$
\mathsf{K}^2_{\alpha_1'}(\kappa) = \sum_{1\leqslant\mu_1^1<\kappa}\frac{1}{\kappa}\cdot\underline{(\mu_1^1-1)!}_{\circ}\cdot\frac{1}{\underline{(\mu_1^1-1)!}_{\circ}}\cdot\big[\log(\kappa)-\log(\mu_1^1)\big]^{\alpha_1'},
$$

and is seen to be an approximation of the Riemann integral:

$$\int_0^1 \big(-\log(x)\big)^{\alpha_1'} = \alpha_1'!,$$

which is finite. In dimensions $n=3$ and $n=4$, the sum writes:

$$\begin{aligned}\mathsf{K}^3_{\alpha_1',\alpha_2'}(\kappa) = \sum_{\substack{1\leqslant\mu_1^1<\kappa\\ 1\leqslant\mu_1^2<\mu_2^2<\kappa}}\ \sum_{\mu_1^1\geqslant\mu_1^2} \frac{1}{\kappa^3}\cdot\kappa!\cdot(\mu_1^1-1)!\cdot\frac{(\mu_1^1-\mu_1^2+\kappa-\mu_2^2)!}{(\mu_1^1-\mu_1^2)!\,(\kappa-\mu_2^2)!}\,\frac{(\mu_2^2+\mu_1^2-1)!}{(\kappa+\mu_1^1)!}\cdot\\ \cdot\frac{1}{(\mu_1^2-1)!\,(\mu_2^2-2)!}\cdot\big[\log\kappa-\log\mu_1^1\big]^{\alpha_1'}\big[\log(\kappa+\mu_1^1)-\log(\mu_2^2+\mu_1^2)\big]^{\alpha_2'},\end{aligned}$$

and:

$$\begin{aligned}\mathsf{K}^4_{\alpha_1',\alpha_2',\alpha_3'}(\kappa) = \sum_{\substack{1\leqslant\mu_1^1<\kappa\\ 1\leqslant\mu_1^2<\mu_2^2<\kappa\\ 1\leqslant\mu_1^3<\mu_2^3<\mu_3^3<\kappa}}\ \sum_{\substack{\mu_1^1\geqslant\mu_1^2\geqslant\mu_1^3\\ \mu_2^2\geqslant\mu_2^3}} \frac{1}{\kappa^6}\cdot(\kappa!)^2\cdot(\mu_1^1-1)!\cdot\frac{(\mu_1^1-\mu_1^2+\kappa-\mu_2^2)!}{(\mu_1^1-\mu_1^2)!\,(\kappa-\mu_2^2)!}\,\frac{(\mu_2^2+\mu_1^2-1)!}{(\kappa+\mu_1^1)!}\cdot\\ \cdot\frac{(\mu_1^2-\mu_1^3+\mu_2^2-\mu_2^3+\kappa-\mu_3^3)!}{(\mu_1^2-\mu_1^3)!\,(\mu_2^2-\mu_2^3)!\,(\kappa-\mu_3^3)!}\,\frac{(\mu_3^3+\mu_2^3+\mu_1^3-1)!}{(\kappa+\mu_2^2+\mu_1^2)!}\cdot\\ \cdot\frac{1}{(\mu_1^3-1)!\,(\mu_2^3-2)!\,(\mu_3^3-3)!}\cdot\\ \cdot\big[\log\kappa-\log\mu_1^1\big]^{\alpha_1'}\big[\log(\kappa+\mu_1^1)-\log(\mu_2^2+\mu_1^2)\big]^{\alpha_2'}\\ \cdot\big[\log(\kappa+\mu_2^2+\mu_1^2)-\log(\mu_3^3+\mu_2^3+\mu_1^3)\big]^{\alpha_3'}.\end{aligned}$$

When $n=3$, we have:

$$\sum_{\substack{1\leqslant\mu_1^1<\kappa\\ 1\leqslant\mu_1^2<\mu_2^2<\kappa}}\ \sum_{\mu_1^1\geqslant\mu_1^2} \equiv \sum_{\mu_1^2=1}^{\kappa-1}\Bigg(\sum_{\mu_2^2=\mu_1^2+1}^{\kappa-1}\Bigg(\sum_{\mu_1^1=\mu_1^2}^{\kappa-1}\bullet\Bigg)\Bigg).$$

When $\alpha_1'=\alpha_2'=0$, the first summation $\sum_{\mu_1^1=\mu_1^2}^{\kappa-1}$ disregarding the $\frac{1}{\kappa^3}$ gives:

$$\begin{aligned}&\sum_{\mu_1^1=\mu_1^2}^{\kappa-1}\kappa!\cdot(\mu_1^1-1)!\cdot\frac{(\mu_1^1-\mu_1^2+\kappa-\mu_2^2)!}{(\mu_1^1-\mu_1^2)!\,(\kappa-\mu_2^2)!}\,\frac{(\mu_2^2+\mu_1^2-1)!}{(\kappa+\mu_1^1)!}\cdot\frac{1}{(\mu_1^2-1)!\,(\mu_2^2-2)!}=\\ &=(c-1)-\frac{(\kappa-1)!\,(2\kappa-\mu_1^2-\mu_2^2)!\,(\mu_1^2+\mu_2^2-1)!\,\kappa!}{(2\kappa)!\,(\kappa-\mu_2^2)!\,(\kappa-\mu_1^2)!\,(\mu_1^2-1)!\,(\mu_2^2-2)!}\sum_{l=0}^{\infty}\frac{(k)_l\,(2k+1-\mu_1^2-\mu_2^2)_l}{(2\kappa+1)_l\,(\kappa-\mu_1^2+1)},\end{aligned}$$

where $(j)_l := j(j+1)\cdots(j+l-1) = \frac{(j+l-1)!}{j!}$. The sum being positive, the absolute value of the second negative term is necessarily $<(c-1)$. But then:

$$\frac{1}{\kappa^3}\sum_{\mu_1^2=1}^{\kappa-1}\Bigg(\sum_{\mu_2^2=\mu_1^2+1}^{\kappa-1}(c-1)\Bigg)=\frac{1}{k^3}\left[\tfrac{1}{3}\,k^3-\tfrac{3}{2}\,k^2+\tfrac{13}{6}\,k-1\right]$$

is clearly bounded independently of κ. So $\mathsf{K}^3_{0,0}(\kappa)\leqslant\mathsf{Constant}_3$ in any case, and so on.

Indirect majorations. Looking back at the Euler-Poincaré characteristic, in the summation formula:

$$\chi\big(X,\, \mathsf{Gr}^\bullet \mathscr{E}_{\kappa,m}^{GG} T_X^*\big) = \sum_{\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n \geqslant 0} M_{\ell_1,\ell_2,\cdots,\ell_n} \cdot \chi\big(\big(X,\, \mathscr{S}^{(\ell_1,\ell_2,\ldots,\ell_n)} T_X^*\big),$$

the coefficients of each Chern monomial $\mathsf{c}_1^{\lambda_1} c_2^{\lambda_2} \cdots \mathsf{c}_n^{\lambda_n}$ must identify. In the Euler-Poincaré characteristic of the Schur bundle, the coefficient of c_1^n is, up to a rational factor:

$$\prod_{1 \leqslant i < j \leqslant n} (\ell_i - \ell_j) \sum_{\beta_1 + \cdots + \beta_{n-1} + \beta_n = n} \ell_1^{\beta_1} \cdots \ell_{n-1}^{\beta_{n-1}} \ell_n^{\beta_n}.$$

We then rewrite:

$$\sum_{\beta_1 + \cdots + \beta_{n-1} + \beta_n = n} \ell_1^{\beta_1} \cdots \ell_{n-1}^{\beta_{n-1}} \ell_n^{\beta_n} = \sum_{\beta_1' + \cdots + \beta_{n-1}' + \beta_n'} C_{\beta_1',\ldots,\beta_{n-1}',\beta_n'} \, (\ell_1 - \ell_2)^{\beta_1'} \cdots (\ell_{n-1} - \ell_n)^{\beta_{n-1}'} \ell_n^{\beta_n'},$$

with coefficients $C_{\beta_1',\ldots,\beta_{n-1}',\beta_n'} \in \mathbb{N}$. Notice that $C_{0,\ldots,0,n} = \binom{n+n-1}{n}$. Identifying then the coefficients of c_1^n:

$$\boxed{\begin{aligned} \sum_{\substack{\text{YT semi-standard} \\ \text{weight(YT)}=m}} \prod_{1 \leqslant i < j \leqslant n} (\ell_i - \ell_j) \cdot (\ell_n)^n = \tfrac{m^{(\kappa+1)n-1}}{((\kappa+1)n-1)!} \, 1!\, 2! \cdots (n-1)! \, (\log \kappa)^n + \\ + \mathsf{O}_n\big(m^{(\kappa+1)n-1} \cdot (\log \kappa)^{n-1}\big) + \mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big). \end{aligned}}$$

The power $(\ell_n)^n$ of ℓ_n corresponds to $(\log \kappa)^n$.

Similarly, in dimension $n = 2$, looking at the coefficient of c_2 and making identification, one gets:

$$\sum_{1 \leqslant \lambda \leqslant \kappa} \frac{1}{\lambda^2} = \sum_{1 \leqslant \mu_1^1 < \kappa} (\kappa!)^2 \, \frac{N_{\mu_1^1}^{\kappa}}{\kappa \cdots \mu_1^1} \, \frac{N_{1,2}^{\mu_1^1,\kappa}}{(\kappa+\mu_1^1) \cdots (2+1)} \left(\sum_{q_0^1 + \cdots + q_{\tau^1}^1 = 3} \frac{1}{(\mu_1^1)^{q_0^1} \cdots (\kappa)^{q_{\tau^1}^1}} \right),$$

so one deduces without computation that the sum:

$$\sum_{1 \leqslant \mu_1^1 < \kappa} (\kappa!)^2 \, \frac{N_{\mu_1^1}^{\kappa}}{\kappa \cdots \mu_1^1} \, \frac{N_{1,2}^{\mu_1^1,\kappa}}{(\kappa+\mu_1^1) \cdots (2+1)} \, \big[\log \kappa - \log \mu_1^1 \big]^3$$

is finite and bounded independently of κ. In dimensions $n = 3$ and higher, looking at the coefficient of c_n, one sees indirectly, without computations and without majorations that all the sums $\square_{n,\kappa}^{\alpha_1',\ldots,\alpha_{n-1}',0}$ which appear after expressing:

$$\prod_{1 \leqslant i < j \leqslant n} \sum_{\beta_1' + \cdots + \beta_{n-1}' = n} \ell_1^{\beta_1'} \cdots \ell_{n-1}^{\beta_{n-1}'}$$

in terms of $(\ell_1 - \ell_2), \ldots, (\ell_{n-1} - \ell_n)$, ℓ_n are finite. This suffices for our purposes.

Summary. In conclusion, either directly or indirectly by identification without computations and without majorations, we have seen that for any $\alpha_1', \dots, \alpha_{n-1}', \alpha_n' \in \mathbb{N}$ with $\alpha_1' + \cdots + \alpha_{n-1}' + \alpha_n' \leqslant \frac{n(n+1)}{2}$, the quantity $\Delta_{n,\kappa}^{\alpha_1', \dots, \alpha_{n-1}', 0}$ is $\leqslant \mathsf{Constant}_n$, whence $\square_{n,\kappa}^{\alpha_1', \dots, \alpha_n'}$ is $\leqslant \mathsf{Constant}_n \cdot (\log \kappa)^{\alpha_n'}$, and from (24), it follows at the end that:

$$\begin{aligned} \sum_{\substack{\mathsf{YT\ semi-standard} \\ \mathsf{weight(YT)}=m}} & \big(\ell_1(\mathsf{YT}) - \ell_2(\mathsf{YT})\big)^{\alpha_1'} \cdots \big(\ell_{n-1}(\mathsf{YT}) - \ell_n(\mathsf{YT})\big)^{\alpha_{n-1}'} \leqslant \\ & \leqslant \frac{m^{(\kappa+1)n-1}}{((\kappa+1)n-1)!\,(\kappa!)^n} \cdot \mathsf{Constant}_n + \mathsf{Constant}_{n,\kappa} \cdot m^{(\kappa+1)n-2}. \end{aligned} \tag{27}$$

§11. Algebraic sheaf theory and Schur bundles

Complex projective hypersurface and line bundles $\mathscr{O}_X(k)$. Let $X = X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a geometrically smooth complex projective hypersurface of degree $d \geqslant 1$, defined in homogeneous coordinates $z = [z_0 \colon z_1 \colon \cdots \colon z_n \colon z_{n+1}]$ as the zero-set:

$$X = \big\{[z_0 \colon z_1 \colon \cdots \colon z_n \colon z_{n+1}] \in \mathbb{P}^{n+1}(\mathbb{C}) \colon P(z_0, z_1, \dots, z_n, z_{n+1}) = 0\big\}$$

of a certain holomorphic polynomial $P = P(z) \in \mathbb{C}[z_0, z_1, \dots, z_n, z_{n+1}]$ which is homogeneous of a certain degree $d \geqslant 1$ and whose differential $P_{z_0} dz_0 + \cdots + P_{z_{n+1}} dz_{n+1}$ does not vanish at any point of X, so that X has no singularities. We will sometimes use the letter N to denote $n+1$:

$$N \overset{\mathsf{notation}}{\equiv} n + 1.$$

The *tautological line bundle* over $\mathbb{P}^N$ will be denoted by $\mathscr{O}_{\mathbb{P}^N}(-1)$ and its dual by $\mathscr{O}_{\mathbb{P}^N}(1) := \mathscr{O}_{\mathbb{P}^N}(-1)^*$. For various values of the integer $k \in \mathbb{Z}$, the standard line bundles:

$$\mathscr{O}_{\mathbb{P}^N}(k) := \mathscr{O}_{\mathbb{P}^N}(\pm 1)^{\otimes |k|},$$

where $\pm = \mathrm{sign}(k)$, will play a very decisive rôle in what follows, as well as their restrictions to X, namely the bundles:

$$\mathscr{O}_X(k) := \mathscr{O}_{\mathbb{P}^N}(k)\big|_X.$$

Canonical line bundles. For any $\mathbb{P}^N$, the (line) bundle of holomorphic differential forms of maximal degree N on $\mathbb{P}^N$:

$$K_{\mathbb{P}^N} = \Lambda^N T^*_{\mathbb{P}^N} \simeq \mathscr{O}_{\mathbb{P}^N}(-N-1),$$

is known, thanks to the *adjunction formula*, to be isomorphic to $\mathscr{O}_{\mathbb{P}^N}(-N-1)$. Similarly, the (line) bundle of holomorphic differential forms of maximal degree n on X:

$$K_X \overset{\mathsf{notation}}{\equiv} \Lambda^n T^*_X \simeq \mathscr{O}_X(d-n-2)$$

called the *canonical bundle* of X and central in complex algebraic geometry, is known, again thanks to the adjunction formula, to be isomorphic to $\mathscr{O}_X(d-n-2)$.

Normal exact sequence. To begin with, one has the so-called *normal exact sequence*:

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^{n+1}}(-d) \overset{\textsf{\textit{incl}}}{\longrightarrow} \mathscr{O}_{\mathbb{P}^{n+1}}(0) \overset{\textsf{\textit{rest}}}{\longrightarrow} \mathscr{O}_X(0) \longrightarrow 0. \tag{28}$$

Here, the inclusion *incl* is defined by multiplication with the defining polynomial $P(z_0, \dots, z_{n+1})$ for X, and the restriction *rest*, of course from $\mathbb{P}^{n+1}$ to X, concerns functions, differential forms, bundles and sheaves.

General sheaves of differential forms. Let r be an integer with $0 \leqslant r \leqslant n+1$ and consider the bundle $\Lambda^r T_X^*$ of differential forms of degree r on X, with the convention that:

$$\Lambda^0 T_X^* \overset{\textsf{collapse}}{\equiv} \mathscr{O}_X(0). \tag{29}$$

The functor $\mathscr{F} \longmapsto \mathscr{F} \otimes \mathscr{G}$ is right exact, for any sheaf $\mathscr{G}$, and is furthermore also left exact when $\mathscr{G}$ is locally free (in what follows, only such sheaves will be considered). Here at any point $z \in X$, the bundle $\Lambda^k T_X^*$ is, for any k with $0 \leqslant k \leqslant n$, a free $\mathscr{O}_{X,z}$-module of rank $\binom{n}{k}$, hence by tensoring the above normal exact sequence, we obtain the following exact sequence:

$$\begin{aligned} 0 \longrightarrow \Lambda^k T^*_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(-d) \longrightarrow \Lambda^k T^*_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(0) \longrightarrow& \\ \longrightarrow \Lambda^k T^*_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_X(0) \longrightarrow 0.& \end{aligned}$$

Hook lengths of Young diagrams. More generally, let $r \geqslant 1$ be any nonnegative integer and let:

$$(\ell) = (\ell_1, \ell_2, \dots, \ell_n)$$

be an arbitrary *partition* of r in at most n parts, namely the sum:

$$\ell_1 + \ell_2 + \cdots + \ell_n = r$$

equals r, and the *parts* ℓ_i are ordered decreasingly:

$$\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n \geqslant 0.$$

Let $d_1, d_2, \dots, d_{\ell_1}$ denote the column lengths of the diagram consisting of ℓ_1 blank squares above ℓ_2 blank squares, ..., above ℓ_n blank squares.
With a bit more precisions, we hence can denote our arbitrary partition as:

$$\left[\begin{aligned} &(\ell) = \big(\ell_1, \ell_2, \dots, \ell_{d_1}, 0, \dots, 0\big) \\ &\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_{d_1} \geqslant 1, \end{aligned}\right.$$

and as in Section 4, we will denote by:

$$\mathsf{YD}_{(\ell)} = \mathsf{YD}_{(\ell_1, \ell_2, \dots, \ell_{d_1}, 0, \dots, 0)}$$

the associated Young diagram. The *hook-length* $h_{i,j}$ of the diagram at the square of coordinates (i, j) is equal to:

$$h_{i,j} := \ell_i - j + d_j - i + 1.$$

A preliminary combinatorial fact, useful soon, is as follows.

Theorem. ([19]) *The number of ways to fill in the r blank cases of the diagram* $\mathsf{YD}_{(\ell_1,\dots,\ell_n)}$ *just with the first r nonnegative integers $1, 2, 3, \dots, r$ in such a way that the appearing integers do increase (strictly) along each row and do also increase (strictly) along each column is equal to the* integer:

$$\nu_{(\ell)} := \frac{r!}{\prod_{i,j} h_{i,j}}.$$

Schur bundles. On every fiber $\big(T^*_{X,x}\big)^{\otimes r}$ of the r-th tensor bundle $\big(T^*_X\big)^{\otimes r}$ over a point $x \in X$, the full linear group $\mathsf{GL}_n(\mathbb{C}) \ni \mathsf{w}$ acts in a natural way:

$$\mathsf{w} \cdot v^*_{i_1} \otimes v^*_{i_2} \otimes \cdots \otimes v^*_{i_r} := \mathsf{w}(v^*_{i_1}) \otimes \mathsf{w}(v^*_{i_2}) \otimes \cdots \otimes \mathsf{w}(v^*_{i_r}),$$

if by $(v^*_1, v^*_2, \dots, v^*_n)$ one denotes any fixed basis of $T^*_{X,x}$. Since the works of Schur?? at the end of the 19$^{\text{th}}$, it is known (*see* [19]) how one may decompose this action into irreducible (nondecomposable) representations which generate the Schur bundles $\mathscr{S}^{(\ell_1,\ell_2,\dots,\ell_n)} T^*_X$ that were already considered in Section 4. Let us provide more information here.

A *Young tableau* $\mathsf{YT}_{1,2,\dots,r}$ is a filling of a given Young diagram $\mathsf{YD}_{(\ell_1,\dots,\ell_n)}$ having $r = \ell_1 + \cdots + \ell_n$ blank boxes precisely by means of the first r positive integers $1, 2, \dots, r$. Notice *passim* that only a special kind of Young tableaux was considered in the theorem above, namely those which enjoy strict increase both along lines and columns, and such combinatorial objects are usually called *standard Young tableaux*.

Idempotents in the group algebra of permutations. Introduce also the *group algebra* $\mathbb{Q} \cdot \mathfrak{S}_r$ over the permutation group:

$$\mathfrak{S}_r = \mathsf{Perm}\big(\{1, 2, \dots, r\}\big),$$

whose general element is a typical sum $\sum_{\sigma \in \mathfrak{S}_r} c_\sigma \cdot \sigma$ having arbitrary rational coefficients $c_\sigma \in \mathbb{Q}$, the addition:

$$\textstyle\sum_{\sigma \in \mathfrak{S}_r} c_\sigma \cdot \sigma + \sum_{\sigma \in \mathfrak{S}_r} d_\sigma \cdot \sigma = \displaystyle\sum_{\sigma \in \mathfrak{S}_r} \big(c_\sigma + d_\sigma\big) \cdot \sigma$$

being obvious and the "multiplication":

$$\Big(\textstyle\sum_{\sigma' \in \mathfrak{S}_r} c_{\sigma'} \cdot \sigma'\Big) \circ \Big(\sum_{\sigma'' \in \mathfrak{S}_r} c_{\sigma''} \cdot \sigma''\Big) = \displaystyle\sum_{\sigma' \in \mathfrak{S}_r} \sum_{\sigma'' \in \mathfrak{S}_r} c_{\sigma'}\, c_{\sigma''} \cdot \sigma' \circ \sigma''$$

corresponding naturally to the composition $\sigma' \circ \sigma''$ of permutations. For a given Young tableau $\mathsf{YT}_{1,\dots,r}$ which shall also be denoted shortly by T, one introduces the following element:

$$e_{\mathsf{T}} := \frac{\nu_{(\ell)}}{r!} \cdot \bigg(\sum_{q \in Q_{\mathsf{T}}} \mathsf{sgn}(q) \cdot q\bigg) \circ \bigg(\sum_{p \in P_{\mathsf{T}}} p\bigg) \tag{30}$$

of the group algebra $\mathbb{Q} \cdot \mathfrak{S}_r$, where Q_{T} denotes the set of permutations that preserve the numbers present in each column of T, and where similarly P_{T} denotes the set of permutations that preserve the numbers present in each row of T.

Theorem. ([19]) *This element e_{T} is an idempotent:*

$$e_{\mathsf{T}} \circ e_{\mathsf{T}} = e_{\mathsf{T}},$$

and the identity permutation $\mathsf{Id} \in \mathbb{Q} \cdot \mathfrak{S}_r$ decomposes as the sum of all such idempotents:

$$\mathsf{Id} = \sum_{\substack{\mathsf{T}=\text{Young tableau} \\ \mathsf{Card}(\mathsf{T})=r}} e_{\mathsf{T}}.$$

Canonical decomposition of tensor powers of the cotangent bundle. The symmetric group $\mathfrak{S}_r$ and therefore also the group algebra $\mathbb{Q} \cdot \mathfrak{S}_r$, act on $\big(T_X^*\big)^{\otimes r}$ just by permuting the spots inside the tensor product:

$$\sigma \cdot v_1 \otimes v_2 \otimes \cdots \otimes v_r := v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(r)}.$$

The identity decomposition (30) then yields at any point $x \in X$ the direct sum decomposition of the r-th tensor power of the cotangent space:

$$\big(T_{X,x}^*\big)^{\otimes r} = \bigoplus_{\substack{\mathsf{T}=\text{Young tableau} \\ \mathsf{Card}(\mathsf{T})=r}} = \mathscr{S}^{\mathsf{T}} T_{X,x}^* \qquad \text{with} \qquad \mathscr{S}^{\mathsf{T}} T_{X,x}^* := e_{\mathsf{T}} \cdot \big(T_{X,x}^*\big)^{\otimes r}.$$

This generalizes the simple well known case $r = 2$:

$$\big(T_{X,x}\big)^{\otimes 2} = \Lambda^2 T_{X,x}^* \oplus \mathrm{Sym}^2 T_{X,x}^*.$$

Theorem. *For any Young Tableau T, a basis of $\mathscr{S}^{\mathsf{T}} T_{X,x}^*$ as a $\mathbb{C}$-vector space is constituted of all vectors of the form:*

$$e_{\mathsf{T}}\big(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r}\big),$$

for any choice of integers $i_1, i_2, \dots, i_r \in \{1, \dots, n\}$ having the property that the filling of the blank boxes of the underlying Young diagram with the integers $i_1, \dots, i_{\ell_1}$ in the first line, then with the integers $i_{\ell_1+1}, \dots, i_{\ell_1+\ell_2}$ in the second line, and so on, provides at the end a semi-standard Young tableau, in the sense that integers are always nondecreasing when read in each row from left to right, and are always increasing (strictly) when read in each column from top to bottom.

It turns out ([19, 6, 18, 11, 35]) that, if two arbitrary Young tableaux T and $\widetilde{\mathsf{T}}$ correspond to the same Young diagram, *i.e.* to the same partition, then $\mathscr{S}^{\mathsf{T}} T_{X,x}^*$ and $\mathscr{S}^{\widetilde{\mathsf{T}}} T_{X,x}^*$ are *isomorphic*. Moreover, for any T, the linear action of $\mathsf{GL}_n(\mathbb{C})$ being compatible with the changes of chart on X and on T_X^*, one may show that the various fibers $\mathscr{S}^{\mathsf{T}} T_{X,x}^*$ organize coherently as a holomorphic bundle over X.

In conclusion, a fundamental Schur bundle decomposition theorem holds which gives the complete generalization of, say:

$$\begin{aligned}\big(T_X^*\big)^{\otimes 2} &= \mathscr{S}^{(2,0,\dots,0)}T_X^* \oplus \mathscr{S}^{(1,1,0,\dots,0)}T_X^*,\\ \big(T_X^*\big)^{\otimes 3} &= \mathscr{S}^{(3,0,\dots,0)}T_X^* \oplus \big[\mathscr{S}^{(2,1,0,\dots,0)}T_X^*\big]^{\oplus 2} \oplus \mathscr{S}^{(1,1,1,0,\dots,0)}T_X^*,\end{aligned}$$

provided X is of dimension $\geqslant 3$; the last factor is dropped when $\dim X = 2$.

Theorem. *For any integer $r \geqslant 1$, the r-th tensor power of the cotangent bundle T_X^* of an arbitrary n-dimensional complex manifold X splits up as a direct sum of so-called Schur bundles:*

$$\big(T_X^*\big)^{\otimes r} = \bigoplus_{\substack{\ell_1\geqslant\ell_2\geqslant\cdots\geqslant\ell_n\geqslant 0\\ \ell_1+\ell_2+\cdots+\ell_n=r}} \big(\mathscr{S}^{(\ell_1,\ell_2,\dots,\ell_n)}T_X^*\big)^{\oplus\nu(\ell)}$$

indexed by all the partitions (ℓ) of r. The rank of $\mathscr{S}^{(\ell_1,\dots,\ell_n)}T_X^*$ as a complex vector bundle equals:

$$\mathrm{rank}\big(\mathscr{S}^{(\ell_1,\ell_2,\dots,\ell_n)}T_X^*\big) = \prod_{1\leqslant i<j\leqslant n}\left(\frac{\ell_i-\ell_j}{i-j}+1\right),$$

and the integer multiplicities:

$$\nu_{(\ell)} = \frac{r!}{\prod_{i,j} h_{i,j}}$$

appearing in the decomposition are expressible in terms of the hook lengths $h_{i,j}$ of the concerned Young diagram $\mathsf{YD}_{(\ell_1,\ell_2,\dots,\ell_n)}$.

Dividing by K_X. Our main goal will now be to control the cohomology of the $\mathscr{S}^{(\ell_1,\dots,\ell_n)}T_X^*$ by a formula which will complement the inequality of the theorem on p. 37, in the case where ℓ_n is large (whence all the ℓ_i are so too). It is then natural to use the known formula:

$$\mathscr{S}^{(\ell_1,\dots,\ell_n)}T_X^*\otimes K_X = \mathscr{S}^{(\ell_1,\dots,\ell_n)}T_X^*\otimes\mathscr{S}^{(1,\dots,1)}T_X^* = \mathscr{S}^{(\ell_1+1,\dots,\ell_n+1)}T_X^*$$

under the subtraction form:

$$\begin{aligned}\mathscr{S}^{(\ell_1,\dots,\ell_{n-1},\ell_n)}T_X^* &= \mathscr{S}^{(\ell_1-\ell_n,\dots,\ell_{n-1}-\ell_n,0)}T_X^*\otimes(K_X)^{\otimes\ell_n}\\ &= \mathscr{S}^{(\ell_1-\ell_n,\dots,\ell_{n-1}-\ell_n,0)}T_X^*\otimes\mathscr{O}_X\big(\ell_n(d-n-2)\big),\end{aligned}$$

which underlines twisting by a certain $\mathscr{O}_X(t)$. On the occasion, it is known thanks to analytical tools (cf. Section 6 in [12]) that if $\mathscr{E}$ is *any* holomorphic vector bundle on the hypersurface $X\subset\mathbb{P}^{n+1}(\mathbb{C})$ and if $\mathscr{L}$ is an ample (or even nef) *line* bundle on X, then:

$$\dim H^q\big(X,\ \mathscr{E}\otimes\mathscr{L}^{\otimes k}\big) = \mathsf{O}(k^{n-q}),$$

for any $q = 0,1,2,\dots,n$. Using purely algebraic tools, what we will do now is to make this estimate much more effective in the case we are interested in,

namely when $\mathscr{E} = \mathscr{S}^{(\ell_1-\ell_n,\dots,\ell_{n-1}-\ell_n,0)}T_X^*$ and when $\mathscr{L} = \mathscr{O}_X(1)$ on a general type hypersurface $X \subset \mathbb{P}^{n+1}$; in this case, X is of degree $d \geqslant n+3$, whence:

$$K_X = \mathscr{O}_X(d-n-2) = \mathscr{O}_X(1)^{\otimes(d-n-2)}$$

is ample of course, so that the exponent $k := \ell_n(d-n-2)$ in:

$$(K_X)^{\ell_n} = \big(\mathscr{O}_X(1)\big)^{\otimes(\ell_n(d-n-2))} = \mathscr{L}^{\otimes(\ell_n(d-n-2))}$$

is positive and in fact large. However, the Landau-type estimate "O" above provided by analytic techniques is not precise enough and we need instead explicit *inequalities*. To achieve more effective estimates, three fundamental exact sequences of holomorphic vector bundles due to Lascoux ([23]) and to Brückmann ([6]) will be very helpful. Thus our goal is to study the cohomology of the twisted Schur bundles:

$$\mathscr{S}^{(\ell_1',\dots,\ell_{n-1}',0)}T_X^* \otimes \mathscr{O}_X(t),$$

when t is large.

First fundamental (long) exact sequence. Dualizing the Euler exact sequence:

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^{n+1}}(0) \longrightarrow \mathscr{O}_{\mathbb{P}^{n+1}}(1)^{\oplus(n+2)} \longrightarrow T_{\mathbb{P}^{n+1}} \longrightarrow 0,$$

we get as a starter the exact sequence:

$$0 \longrightarrow T_{\mathbb{P}^{n+1}}^* \longrightarrow \mathscr{O}_{\mathbb{P}^{n+1}}(-1)^{\oplus(n+2)} \longrightarrow \mathscr{O}_{\mathbb{P}^{n+1}}(0) \longrightarrow 0.$$

The procedure explained by Brückmann in [6] of taking the r-th tensor power of the extracted complex composed of the last two bundles:

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathscr{O}_{\mathbb{P}^{n+1}}(-1)^{\oplus(n+2)} \longrightarrow \mathscr{O}_{\mathbb{P}^{n+1}}(0) \longrightarrow 0$$

and more generally, of taking any of its Schur powers, provides a useful long exact sequence of holomorphic vector bundles which gives a free resolution of $\mathscr{S}^{(\ell_1,\dots,\ell_n,\ell_{n+1})}T_{\mathbb{P}^{n+1}}^*$ on $\mathbb{P}^{n+1}$. Instead of using the same letter $\mathscr{S}$ for Schur bundles over $\mathbb{P}^{n+1}$ and over X, we shall, in order to underline a clearly visible distinction between $\mathbb{P}^{n+1}$ and X, write:

$$\mathscr{S}^{(\ell_1,\dots,\ell_n,\ell_{n+1})}T_{\mathbb{P}^{n+1}}^* \overset{\mathsf{notation}}{\equiv} \Omega_{\mathbb{P}^{n+1}}^{(\ell_1,\dots,\ell_n,\ell_{n+1})},$$

using the Greek letter[27] Ω with '$\mathbb{P}^{n+1}$' placed at the lower index place.

Let now T be a Young tableau with r boxes and with row lengths $\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_{n+1} \geqslant 0$, hence of depth $\leqslant n+1$. For convenient abbreviation, we introduce the general notation:

$$\Delta\big(\theta_1, \theta_2, \dots, \theta_K\big) := \prod_{1\leqslant i<j\leqslant K} \big(\theta_i - \theta_j\big)$$

[27] Justification: in several articles, the letter Ω is employed to denote the bundles Ω^k or $\Omega^k T_X^*$, $0 \leqslant k \leqslant n$, that we denoted by $\Lambda^k T_X^*$ above.

which is, up to sign, the value:

$$\begin{vmatrix} 1 & \theta_1 & \theta_1^2 & \cdots & \theta_1^{K-1} \\ 1 & \theta_2 & \theta_2^2 & \cdots & \theta_2^{K-1} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot \\ 1 & \theta_K & \theta_K^2 & \cdots & \theta_K^{K-1} \end{vmatrix} = (-1)^{\frac{K(K-1)}{2}} \Delta\big(\theta_1, \theta_2, \dots, \theta_K\big)$$

of a corresponding Van der Monde determinant.

Theorem. ([6]) *Let $d_1 = \mathrm{depth}(\mathsf{T})$ be the depth of T, which is $\leqslant n+1$, let $r = \ell_1 + \cdots + \ell_{d_1}$ be the number of boxes of T, set:*

$$t_i := r + \ell_i - i$$

for all $i = 1, 2, \dots, n+1, n+2$ with of course:

$$t_{d_1+1} = r - d_1 - 1, \ \dots\dots, \ t_{n+1} = r - n - 1, \ \ t_{n+2} = r - n - 2,$$

and define the rational number:

$$b_0 := \tfrac{1}{1!\,2!\,\cdots\,n!\,(n+1)!} \cdot \Delta\big(t_1, \dots, t_{n+1}, t_{n+2}\big),$$

together with, for any $s = 1, 2, \dots, d_1$, the rational numbers:

$$b_s := \tfrac{1}{1!\,2!\,\cdots\,n!\,(n+1)!} \cdot \sum_{1 \leqslant i_1 < \cdots < i_s \leqslant d_1} \Delta\big(t_1, t_2, \dots, t_{i_1} - 1, \dots, t_{i_s} - 1, \dots, t_{n+1}, t_{n+2}\big).$$

Then there is a long exact sequence of holomorphic vector bundles over $\mathbb{P}^{n+1}$ of the form:

$$\begin{aligned} 0 \longrightarrow \Omega_{\mathbb{P}^{n+1}}^{(\ell_1, \dots, \ell_{d_1}, 0, \dots, 0)} \longrightarrow \bigoplus_{b_0} \mathscr{O}_{\mathbb{P}^{n+1}}(-r) \longrightarrow \bigoplus_{b_1} \mathscr{O}_{\mathbb{P}^{n+1}}(-r+1) \longrightarrow \cdots \\ \cdots \longrightarrow \bigoplus_{b_{d_1}} \mathscr{O}_{\mathbb{P}^{n+1}}(-r+d_1) \longrightarrow 0. \end{aligned}$$

Then tensoring by $\mathscr{O}_{\mathbb{P}^{n+1}}(t)$ with an arbitrary $t \in \mathbb{Z}$, we get the useful:

(31)
$$\begin{aligned} 0 \longrightarrow \Omega_{\mathbb{P}^{n+1}}^{(\ell_1, \dots, \ell_{d_1}, 0, \dots, 0)} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(t) \longrightarrow \bigoplus_{b_0} \mathscr{O}_{\mathbb{P}^{n+1}}(t-r) \longrightarrow \bigoplus_{b_1} \mathscr{O}_{\mathbb{P}^{n+1}}(t-r+1) \longrightarrow \cdots \\ \cdots \longrightarrow \bigoplus_{b_{d_1}} \mathscr{O}_{\mathbb{P}^{n+1}}(t-r+d_1) \longrightarrow 0. \end{aligned}$$

Second fundamental (short) exact sequence. Because any Schur bundle $\Omega_{\mathbb{P}^{n+1}}^{(\ell_1, \dots, \ell_n, \ell_{n+1})}$ over $\mathbb{P}^{n+1}$ is, according to what precedes, a locally free sheaf of $\mathscr{O}_{\mathbb{P}^{n+1}}$-modules of finite rank $\prod_{1 \leqslant i < j \leqslant n+1} \big(\frac{\ell_i - \ell_j}{j-i} + 1\big)$, a tensorisation of the

normal exact sequence (28) on p. 71 yields the general short exact sequence ([5, 35]):

$$
\begin{aligned}
0 \longrightarrow \Omega_{\mathbb{P}^{n+1}}^{(\ell_1,\dots,\ell_n,\ell_{n+1})} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(-d) \longrightarrow \Omega_{\mathbb{P}^{n+1}}^{(\ell_1,\dots,\ell_n,\ell_{n+1})} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(0) \longrightarrow \\
\longrightarrow \Omega_{\mathbb{P}^{n+1}}^{(\ell_1,\dots,\ell_n,\ell_{n+1})} \otimes \mathscr{O}_X(0) \longrightarrow 0.
\end{aligned}
$$

Tensoring in addition again by $\mathscr{O}_{\mathbb{P}^{n+1}}(t)$ where $t \in \mathbb{Z}$ is arbitrary, knowing $\mathscr{O}_X(0) \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(t) = \mathscr{O}_X(t)$, we deduce the general form of this (second, short) exact sequence that will be useful below:

$$
\begin{aligned}
0 \longrightarrow \Omega_{\mathbb{P}^{n+1}}^{(\ell_1,\dots,\ell_n,\ell_{n+1})} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(t-d) \longrightarrow \Omega_{\mathbb{P}^{n+1}}^{(\ell_1,\dots,\ell_n,\ell_{n+1})} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(t) \longrightarrow \\
\longrightarrow \Omega_{\mathbb{P}^{n+1}}^{(\ell_1,\dots,\ell_n,\ell_{n+1})} \otimes \mathscr{O}_X(t) \longrightarrow 0.
\end{aligned}
\tag{32}
$$

Here, we make the convention similar to (29) on p. 71 that when all the ℓ_i are zero, $\Omega_{\mathbb{P}^{n+1}}^{(0,\dots,0,0)}$ identifies to $\mathscr{O}_{\mathbb{P}^{n+1}}(0)$, whence in this case the written exact sequence reduces just to (28) on p. 71, tensored of course by $\mathscr{O}_{\mathbb{P}^{n+1}}(t)$.

Third fundamental exact sequence. Lastly, starting from the cotangential normal exact sequence:

$$
0 \longrightarrow \mathscr{O}_X(-d) \longrightarrow T^*_{\mathbb{P}^{n+1}}|_X \longrightarrow T^*_X \longrightarrow 0,
\tag{33}
$$

(recall that $T^*_{\mathbb{P}^{n+1}}|_X = T^*_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_X(0)$), Brückmann established that the Schur power of the extracted complex:

$$
0 \longrightarrow \mathscr{O}_X(-d) \longrightarrow T^*_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_X(0) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots
$$

provides a free resolution of $\mathscr{S}^{(\ell_1,\dots,\ell_n)} T^*_X$ (Theorem 3 in [6]) which may be written in great details as follows when $\ell_n \geqslant 1$:

$$
\begin{aligned}
0 \longrightarrow \bigoplus_{\substack{\delta_1+\cdots+\delta_n=n \\ \delta_i = 0 \text{ or } 1}} \Omega_{\mathbb{P}^{n+1}}^{(\ell_1,\dots,\ell_n,0)-(\delta_1,\dots,\delta_n,0)} \otimes \mathscr{O}_X(-nd) \longrightarrow \cdots \\
\cdots \longrightarrow \bigoplus_{\substack{\delta_1+\cdots+\delta_n=k \\ \delta_i = 0 \text{ or } 1}} \Omega_{\mathbb{P}^{n+1}}^{(\ell_1,\dots,\ell_n,0)-(\delta_1,\dots,\delta_n,0)} \otimes \mathscr{O}_X(-kd) \longrightarrow \cdots \\
\cdots \longrightarrow \Omega_{\mathbb{P}^{n+1}}^{(\ell_1,\dots,\ell_n,0)} \otimes \mathscr{O}_X(0) \longrightarrow \mathscr{S}^{(\ell_1,\dots,\ell_n)} T^*_X \longrightarrow 0.
\end{aligned}
$$

Notice that the last upper entry ℓ_{n+1} of each Ω is zero. Of course, the direct sum for the first entry reduces just to the single term:

$$
\Omega_{\mathbb{P}^{n+1}}^{(\ell_1-1,\dots,\ell_n-1,0)} \otimes \mathscr{O}_X(-nd).
$$

In full generality, if d_1 denotes the depth of the considered Young diagram, hence if one has $\ell_1 \geqslant \cdots \geqslant \ell_{d_1} \geqslant 1$ but $0 = \ell_{d_1+1} = \cdots = \ell_n = \ell_{n+1}$, the locally free

resolution of $\mathscr{S}^{(\ell_1,\dots,\ell_{d_1},0,\dots,0)}T_X^*$ reads ([6]):

$$0 \longrightarrow \bigoplus_{\substack{\delta_1+\cdots+\delta_{d_1}=d_1\\ \delta_i=0 \text{ or } 1}} \Omega_{\mathbb{P}^{n+1}}^{(\ell_1,\dots,\ell_{d_1},0,\dots,0,0)-(\delta_1,\dots,\delta_{d_1},0,\dots,0,0)} \otimes \mathscr{O}_X(-d_1 d) \longrightarrow \cdots$$

$$\cdots \longrightarrow \bigoplus_{\substack{\delta_1+\cdots+\delta_{d_1}=k\\ \delta_i=0 \text{ or } 1}} \Omega_{\mathbb{P}^{n+1}}^{(\ell_1,\dots,\ell_{d_1},0,\dots,0,0)-(\delta_1,\dots,\delta_{d_1},0,\dots,0,0)} \otimes \mathscr{O}_X(-kd) \longrightarrow \cdots$$

$$\cdots \longrightarrow \Omega_{\mathbb{P}^{n+1}}^{(\ell_1,\dots,\ell_{d_1},0,\dots,0,0)} \otimes \mathscr{O}_X(0) \longrightarrow \mathscr{S}^{(\ell_1,\dots,\ell_{d_1},0,\dots,0)}T_X^* \longrightarrow 0,$$

hence it just looks like a truncation of the preceding resolution valid when $d_1 = n$. Tensoring this by $\mathscr{O}_X(t)$, we finally get what will be useful below:

(34)

$$0 \longrightarrow \bigoplus_{\substack{\delta_1+\cdots+\delta_{d_1}=d_1\\ \delta_i=0 \text{ or } 1}} \Omega_{\mathbb{P}^{n+1}}^{(\ell_1,\dots,\ell_{d_1},0,\dots,0,0)-(\delta_1,\dots,\delta_{d_1},0,\dots,0,0)} \otimes \mathscr{O}_X(t-d_1 d) \longrightarrow \cdots$$

$$\cdots \longrightarrow \bigoplus_{\substack{\delta_1+\cdots+\delta_{d_1}=k\\ \delta_i=0 \text{ or } 1}} \Omega_{\mathbb{P}^{n+1}}^{(\ell_1,\dots,\ell_{d_1},0,\dots,0,0)-(\delta_1,\dots,\delta_{d_1},0,\dots,0,0)} \otimes \mathscr{O}_X(t-kd) \longrightarrow \cdots$$

$$\cdots \longrightarrow \Omega_{\mathbb{P}^{n+1}}^{(\ell_1,\dots,\ell_{d_1},0,\dots,0,0)} \otimes \mathscr{O}_X(t) \longrightarrow \mathscr{S}^{(\ell_1,\dots,\ell_{d_1},0,\dots,0)}T_X^* \otimes \mathscr{O}_X(t) \longrightarrow 0.$$

Cohomology of Schur bundles over $\mathbb{P}^{n+1}$. In [6] too, using the first exact sequence above plus further arguments, Brückmann established the following theorem which computes completely the dimensions of all the cohomology groups of twisted Schur bundles over $\mathbb{P}^{n+1}$. As above, for fixed $n+1 \geqslant 2$ and for fixed $\ell_1 \geqslant \cdots \geqslant \ell_n \geqslant \ell_{n+1} \geqslant 0$, we introduce the integers:

$$t_i := \ell_i - i + \textstyle\sum_{i=1}^{n+1} \ell_i \qquad\qquad {\scriptstyle (i=1\cdots n,\, n+1)},$$

which, visibly, are ordered decreasingly:

$$t_1 > t_2 > \cdots > t_n > t_{n+1}.$$

Theorem. ([6]) *For any $t \in \mathbb{Z}$, the Euler-Poincaré characteristic of $\Omega_{\mathbb{P}^{n+1}}^{(\ell_1,\dots,\ell_n,\ell_{n+1})} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(t)$ is equal to:*

$$\chi(t) := \chi\big(\mathbb{P}^{n+1},\, \Omega_{\mathbb{P}^{n+1}}^{(\ell_1,\dots,\ell_n,\ell_{n+1})} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(t)\big) = \tfrac{1}{1!\,2!\,\cdots\,n!\,(n+1)!} \prod_{1\leqslant i<j\leqslant n+1} (t_i - t_j) \prod_{1\leqslant i\leqslant n} (t - t_i),$$

whence it vanishes for t equal to each one of the t_i. Furthermore, as t varies in $\mathbb{Z}$, at most one of the cohomology dimensions:

$$h^q(t) := \dim H^q\big(\mathbb{P}^{n+1},\, \Omega_{\mathbb{P}^{n+1}}^{(\ell_1,\dots,\ell_n,\ell_{n+1})} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(t)\big)$$

may be nonzero, and more precisely, $h^q(t)$ is nonzero and equal to $(-1)^q\chi(t)$ if and only if $t_{q+1} + 1 \leqslant t \leqslant t_q - 1$, while the other $h^{q'}(t)$ do vanish for all t in the same range. In particular, for all:

(35) $$t \geqslant \ell_1 + \textstyle\sum_{i=1}^{n+1} \ell_i,$$

all the positive cohomology dimensions vanish:

$$0 = \dim H^q\big(\mathbb{P}^{n+1},\, \Omega_{\mathbb{P}^{n+1}}^{(\ell_1,\dots,\ell_n,\ell_{n+1})} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(t)\big) \qquad\qquad {\scriptstyle (q\,=\,1,\ 2\,\cdots\, n)}.$$

Applications. For an application to the study of the cohomology of Schur bundles over $X^n \subset \mathbb{P}^{n+1}$, we shall apply the above theorems specifically to the Young diagrams $\mathsf{YD}_{(\ell_1,\dots,\ell_n,0)}$ of depth $d_1 \leqslant n = \dim X$, with $\ell_{n+1} = 0$ in order to gain the following complement to the theorem on p. 37.

Theorem. *Let $X = X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a geometrically smooth projective algebraic complex hypersurface of general type, i.e. of degree $d \geqslant n+3$, and let $\ell = (\ell_1,\dots,\ell_{n-1},\ell_n)$ with $\ell_1 \geqslant \cdots \geqslant \ell_{n-1} \geqslant \ell_n \geqslant 1$. If:*

$$\ell_n \geqslant \tfrac{1}{d-n-2}\big\{n(d-1) + \ell_1 - \ell_n + \textstyle\sum_{i=1}^{n-1}(\ell_i - \ell_n)\big\},$$

then all the positive cohomologies vanish:

$$0 = H^q\big(X,\, \mathscr{S}^{(\ell_1,\dots,\ell_{n-1},\ell_n)} T_X^*\big) \qquad\qquad {\scriptstyle (q\,=\,1,\ 2\,\cdots\, n)}.$$

Proof. As anticipated above, after division by $(K_X)^{\ell_n}$, we are lead back to examining the cohomology of:

$$\mathscr{S}^{(\ell_1-\ell_n,\dots,\ell_{n-1}-\ell_n,0)} T_X^* \otimes \mathscr{O}_X\big(\ell_n(d-n-2)\big).$$

A bit more generally, using the second and the third exact sequences (32) and (34), we shall examine when the positive cohomologies of:

$$\mathscr{S}^{(\ell_1',\dots,\ell_{n-1}',0)} T_X^* \otimes \mathscr{O}_X(t')$$

do all vanish, and afterward, we shall set:

$$\ell_1' := \ell_1 - \ell_n,\ \dots,\ \ell_{n-1}' := \ell_{n-1} - \ell_n \quad \text{ and } \quad t' := \ell_n(d-n-2).$$

We assume first that $\ell_{n-1}' \geqslant 1$ and we shall discuss the quite similar case $\ell_{n-1}' = 0$ afterward. The consideration of the third exact sequence (34) with $d_1 = n-1$ and $(\ell_1',\dots,\ell_{n-1}',0)$ instead of $(\ell_1,\dots,\ell_{n-1},0)$ then necessarily conducts us to the study of $\mathscr{O}_X$-twisted Schur bundles over $\mathbb{P}^{n+1}$:

$$\Omega_{\mathbb{P}^{n+1}}^{(\ell_1'',\dots,\ell_{n-1}'',0)} \otimes \mathscr{O}_X(t'')$$

whose Young diagram exponents ℓ_i'' have values:

$$(\ell_1'',\dots,\ell_{n-1}'',0) = (\ell_1',\dots,\ell_{n-1}',0) - (\delta_1',\dots,\delta_{n-1}',0)$$

shifted a bit from the values of the ℓ_i', where $\delta_1' + \cdots + \delta_{n-1}' = k$ for $k = 0,1,\dots,n-1$, with of course $\delta_i' = 0$ or 1. So to begin with, it is advisable to study the cohomology of these $\mathscr{O}_X$-twisted Schur bundles over $\mathbb{P}^{n+1}$.

To this aim, we look at the second (short) exact sequence (32) written with:

$$(\ell_1,\dots,\ell_{n-1},\ell_n,\ell_{n+1}) := (\ell_1'',\dots,\ell_{n-1}'',0,0),$$

for some arbitrary $\ell_1'' \geqslant \cdots \geqslant \ell_{n-1}'' \geqslant 0$ and we abbreviate this exact sequence as:

$$0 \longrightarrow \mathscr{P} \longrightarrow \mathscr{Q} \longrightarrow \mathscr{R} \longrightarrow 0,$$

where $\mathscr{P} \to \mathbb{P}^{n+1}$, $\mathscr{Q} \to \mathbb{P}^{n+1}$ and $\mathscr{R} \to X$ are the bundles:

$$\begin{aligned}\mathscr{P} &:= \Omega_{\mathbb{P}^{n+1}}^{(\ell_1'',\dots,\ell_{n-1}'',0,0)} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(t''-d),\\ \mathscr{Q} &:= \Omega_{\mathbb{P}^{n+1}}^{(\ell_1'',\dots,\ell_{n-1}'',0,0)} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(t''),\\ \mathscr{R} &:= \Omega_{\mathbb{P}^{n+1}}^{(\ell_1'',\dots,\ell_{n-1}'',0,0)} \otimes \mathscr{O}_{X}(t''),\end{aligned}$$

so that all the cohomology dimensions of $\mathscr{P}$ and of $\mathscr{Q}$ are known thanks to Brückmann's theorem on p. 78. Then in the long exact cohomology sequence associated to the short exact sequence:

$$\begin{aligned}0 &\longrightarrow H^0(\mathbb{P}^{n+1},\,\mathscr{P}) \longrightarrow H^0(\mathbb{P}^{n+1},\,\mathscr{Q}) \longrightarrow H^0(X,\,\mathscr{R}) \longrightarrow\\ &\longrightarrow \underline{H^1(\mathbb{P}^{n+1},\,\mathscr{P})}_\circ \longrightarrow \underline{H^1(\mathbb{P}^{n+1},\,\mathscr{Q})}_\circ \longrightarrow H^1(X,\,\mathscr{R}) \longrightarrow\\ &\longrightarrow \underline{H^2(\mathbb{P}^{n+1},\,\mathscr{P})}_\circ \longrightarrow \underline{H^2(\mathbb{P}^{n+1},\,\mathscr{Q})}_\circ \longrightarrow H^2(X,\,\mathscr{R}) \longrightarrow \cdots\\ \cdots &\longrightarrow \underline{H^n(\mathbb{P}^{n+1},\,\mathscr{P})}_\circ \longrightarrow \underline{H^n(\mathbb{P}^{n+1},\,\mathscr{Q})}_\circ \longrightarrow H^n(X,\,\mathscr{R}) \longrightarrow\\ \cdots &\longrightarrow \underline{H^{n+1}(\mathbb{P}^{n+1},\,\mathscr{P})}_\circ \longrightarrow \underline{H^{n+1}(\mathbb{P}^{n+1},\,\mathscr{Q})}_\circ \longrightarrow 0\end{aligned}$$

(the last 0 because $\mathscr{R} \to X$ is a bundle over an n-dimensional basis), all the underlined terms will vanish, namely one will have:

$$\begin{aligned}0 &= H^q\big(\mathbb{P}^{n+1},\ \Omega_{\mathbb{P}^{n+1}}^{(\ell_1'',\dots,\ell_{n-1}'',0,0)} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(t''-d)\big) \qquad {\scriptstyle (q=1,\,2\cdots n,\,n+1)},\\ 0 &= H^q\big(\mathbb{P}^{n+1},\ \Omega_{\mathbb{P}^{n+1}}^{(\ell_1'',\dots,\ell_{n-1}'',0,0)} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(t'')\big) \qquad {\scriptstyle (q=1,\,2\cdots n,\,n+1)},\end{aligned}$$

as soon as the following two inequalities are satisfied by t'':

$$\begin{aligned}t''-d &\geqslant \ell_1'' + \textstyle\sum_{i=1}^{n-1} \ell_i'',\\ t'' &\geqslant \ell_1'' + \textstyle\sum_{i=1}^{n-1} \ell_i'',\end{aligned}$$

as is guaranteed by the inequality (35) of the theorem on p. 78. But the first inequality obviously entails the second one, hence we deduce that all positive cohomologies:

$$0 = H^q\big(X,\ \Omega_{\mathbb{P}^{n+1}}^{(\ell_1'',\dots,\ell_{n-1}'',0,0)} \otimes \mathscr{O}_X(t'')\big) \qquad {\scriptstyle (q=1,\,2\cdots n)}$$

of $\mathscr{R}$ vanish as soon as:

$$t'' \geqslant d + \ell_1'' + \textstyle\sum_{i=1}^{n-1} \ell_i''. \tag{36}$$

We observe that this fact is valid also when $\ell_{n_1+1}'' = \cdots = \ell_{n-1}''$ for some largest integer $n_1 \geqslant 0$ with $\ell_{n_1}'' \geqslant 1$, because the second exact sequence (32) we used is subjected to no restriction.

We now come to dealing with the third exact sequence (34). Cutting a long exact sequence in short exact sequences, one may establish the following standard lemma, used e.g. in [36].

Lemma. *Consider a holomorphic vector bundle $\mathscr{S} \to X$ equipped with a free resolution of length $\leqslant n$ provided by a long exact sequence of holomorphic vector bundles $\mathscr{A}^0, \mathscr{A}^1, \dots, \mathscr{A}^n$ over X:*

$$0 \longrightarrow \mathscr{A}^n \longrightarrow \mathscr{A}^{n-1} \longrightarrow \cdots \longrightarrow \mathscr{A}^1 \longrightarrow \mathscr{A}^0 \longrightarrow \mathscr{S} \longrightarrow 0.$$

Then in order that all the positive cohomology groups vanish:

$$0 = H^1(X,\, \mathscr{S}) = \cdots = H^n(X,\, \mathscr{S}),$$

it suffices that:

$$\begin{aligned}
0 = H^1(X,\, \mathscr{A}^0) = H^2(X,\, \mathscr{A}^1) = H^3(X,\, \mathscr{A}^2) = \cdots = H^n(X,\, \mathscr{A}^{n-1}) \\
0 = H^2(X,\, \mathscr{A}^0) = H^3(X,\, \mathscr{A}^1) = \cdots = H^n(X,\, \mathscr{A}^{n-2}) \\
0 = H^3(X,\, \mathscr{A}^0) = \cdots = H^n(X,\, \mathscr{A}^{n-3}) \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
0 = H^n(X,\, \mathscr{A}^0).
\end{aligned}$$

So as said a short while ago, we aim to apply this lemma when looking at the third exact sequence (34) which, for the case we are interested in, writes precisely under the form:

$$0 \longrightarrow \bigoplus_{\substack{\delta_1'+\cdots+\delta_{n-1}'=n-1 \\ \delta_i' = 0 \text{ or } 1}} \Omega_{\mathbb{P}^{n+1}}^{(\ell_1',\dots,\ell_{n-1}',0,0)-(\delta_1',\dots,\delta_{n-1}',0,0)} \otimes \mathscr{O}_X\big(t'-(n-1)d\big) \longrightarrow \cdots$$

$$\cdots \longrightarrow \bigoplus_{\substack{\delta_1'+\cdots+\delta_{n-1}'=k \\ \delta_i' = 0 \text{ or } 1}} \Omega_{\mathbb{P}^{n+1}}^{(\ell_1',\dots,\ell_{n-1}',0,0)-(\delta_1',\dots,\delta_{n-1}',0,0)} \otimes \mathscr{O}_X\big(t'-kd\big) \longrightarrow \cdots$$

$$\cdots \longrightarrow \Omega_{\mathbb{P}^{n+1}}^{(\ell_1',\dots,\ell_{n-1}',0,0)} \otimes \mathscr{O}_X(t') \longrightarrow \mathscr{S}^{(\ell_1',\dots,\ell_{n-1}',0)} \mathscr{O}_X(t') \longrightarrow 0.$$

In the notations of the lemma, the resolution of:

$$\mathscr{S} := \mathscr{S}^{(\ell_1',\dots,\ell_{n-1}',0)} \otimes \mathscr{O}_X(t')$$

is hence of length $n-1$ when we set:

$$\mathscr{A}^k := \bigoplus_{\substack{\delta_1'+\cdots+\delta_{n-1}'=k \\ \delta_i' = 0 \text{ or } 1}} \Omega_{\mathbb{P}^{n+1}}^{(\ell_1',\dots,\ell_{n-1}',0,0)-(\delta_1',\dots,\delta_{n-1}',0,0)} \otimes \mathscr{O}_X\big(t'-kd\big)$$

$$(k = 0,\, 1 \cdots n-1).$$

Then for the lemma to yield the vanishing of all the positive cohomologies of $\mathscr{S} = \mathscr{S}^{(\ell_1',\dots,\ell_{n-1}',0)} \otimes \mathscr{O}_X(t')$, it is evidently sufficient that plainly all positive cohomologies of the $\mathscr{A}^k$ vanish:

$$0 = H^q(X,\, \mathscr{A}^k) \qquad (q = 1,\, 2 \cdots n;\; k = 0,\, 1 \cdots n-1),$$

which is more than what is required in fact. But since each $\mathscr{A}^k$ is a direct sum, it even suffices that:

$$0 = H^q\big(X,\ \Omega_{\mathbb{P}^{n+1}}^{(\ell_1',\dots,\ell_{n-1}',0,0)-(\delta_1',\dots,\delta_{n-1}',0,0)} \otimes \mathscr{O}_X(t'-kd)\big)$$
$$(q=1,\ 2\cdots n\,;\ \ \delta_1'+\cdots+\delta_{n-1}'=k\,;\ \ k=0,\ 1\cdots n-1).$$

According to (36), this holds true provided all the following inequalities are satisfied:

$$t'-kd \geqslant d+\ell_1'-\delta_1'+\textstyle\sum_{i=1}^{n-1}(\ell_i'-\delta_i'),$$

for every $k=0,1,\dots,n-1$ and every $\delta_1',\dots,\delta_{n-1}'\in\{0,1\}$ with $\delta_1'+\cdots+\delta_{n-1}'=k$. But since $-\delta_i'\leqslant 1$ and since $\sum(-\delta_i')=-k$, it suffices that, firstly for $k=0,1,\dots,n-2$:

$$t'-kd \geqslant d+\ell_1'+\textstyle\sum_{i=1}^{n-1}\ell_i'-k,$$

and secondly for $k=n-1$, whence $-\delta_1'=-1$ necessarily:

$$t'-(n-1)d \geqslant d+\ell_1'-1+\textstyle\sum_{i=1}^{n-1}\ell_i'-(n-1). \tag{37}$$

But this last inequality, rewritten under the form:

$$t' \geqslant n(d-1)+\ell_1'+\textstyle\sum_{i=1}^{n-1}\ell_i'$$

visibly entails all the inequalities for $k=0,1,\dots,n-2$. Lastly, replacing $t'=\ell_n(d-n-2)$ and the $\ell_i'=\ell_i-\ell_n$ by their values, we finally come to the numerical condition claimed by the theorem for positive cohomologies of the $\mathscr{S}^{(\ell_1,\dots,\ell_{n-1},\ell_n)}T_X^*$ to vanish.

To conclude the argument, it only remains to examine what happens with the case, left aside, when $\ell_{n-1}'=0$. In this case, there is a nonnegative integer $n_1\leqslant n-2$ with $\ell_1'\geqslant\cdots\geqslant\ell_{n_1}'\geqslant 1$ while $0=\ell_{n_1+1}'=\cdots=\ell_{n-1}'$. At first, if $n_1=0$, i.e. if all the ℓ_i are equal to ℓ_n, then $\mathscr{S}^{(\ell_n,\dots,\ell_n)}T_X^*=\mathscr{O}_X\big(\ell_n(d-n-2)\big)$ reduces to a standard line bundle $\mathscr{O}_X(t')$, and it is well known that:

$$0=H^q\big(X,\ \mathscr{O}_X(t')\big) \qquad (q=1,\ 2\cdots n)$$

whenever $t'\geqslant 0$.

Therefore, we may assume that n_1 satisfies $1\leqslant n_1\leqslant n-2$. As before, the subtraction of $(K_X)^{\ell_n}$ yields:

$$\ell_1'=\ell_1-\ell_n,\ \dots,\ \ell_{n_1'}=\ell_{n_1}-\ell_n \quad \text{and} \quad 0=\ell_{n_1+1}'=\cdots=\ell_{n-1}'=\ell_n',$$

and again as always $t'=\ell_n(d-n-2)$. In the third exact sequence, the factors then are:

$$\mathscr{A}^k=\bigoplus_{\substack{\delta_1'+\cdots+\delta_{n_1}'=k\\ \delta_i'=0\text{ or }1}}\Omega_{\mathbb{P}^{n+1}}^{(\ell_1',\dots,\ell_{n_1}',0,\dots,0,0)-(\delta_1',\dots,\delta_{n_1}',0,\dots,0,0)}\otimes\mathscr{O}_X(t'-kd)$$
$$(k=0,\ 1\cdots n_1),$$

so the positive cohomologies vanish all provided that:

$$t'-kd\geqslant d+\ell_1'-\delta_1'+\textstyle\sum_{i=1}^{n_1}(\ell_i'-\delta_i') \qquad (k=0,\ 1\cdots n_1\,;\ \ \delta_1'+\cdots+\delta_{n_1}'=n_1),$$

and because $k \leqslant n_1 \leqslant n-2$, these inequalities are all less stringent than the one (37) we found previously in the case when $\ell'_{n-1} \geqslant 1$ (or equivalently, when $n_1 = n-1$). This completes the proof of the theorem. □

§12. Asymptotic cohomology vanishing

Synthesis: uniform majoration for the cohomology of Schur bundles. Two cohomology controls have been achieved. Firstly, according to the theorem stated above on p. 79 and just proved, when:

$$\ell_n \geqslant \tfrac{1}{d-n-2}\left\{n(d-1)+\ell_1-\ell_n+\textstyle\sum_{i=1}^{n-1}(\ell_i-\ell_n)\right\},$$

the positive cohomologies of Schur bundles vanish:

$$0 = h^q\big(X,\, \mathscr{S}^{(\ell_1,\ell_2,\dots,\ell_n)}T_X^*\big) \qquad\qquad {\scriptstyle (q=1,2\cdots n)}.$$

Secondly, according to the theorem stated on p. 37, when:

$$|\ell| \geqslant 1+2n^2+(n+1)(d-n-2),$$

the positive cohomologies enjoy a majoration of the shape:

$$\begin{aligned} h^q\big(X,\, \mathscr{S}^{(\ell_1,\ell_2,\dots,\ell_n)}T_X^*\big) \leqslant \mathsf{Constant}_{n,d}\cdot \prod_{1\leqslant i<j\leqslant n}(\ell_i-\ell_j)\Bigg\{ \\ \Bigg\{\sum_{\beta_1+\cdots+\beta_{n-1}+\beta_n=n}(\ell_1-\ell_2)^{\beta_1}\cdots(\ell_{n-1}-\ell_n)^{\beta_{n-1}}\ell_n^{\beta_n}\Bigg\}+ \\ +\mathsf{Constant}_{n,d}\big(1+|\ell|^{\frac{n(n+1)}{2}-1}\big) \qquad {\scriptstyle (q=1,2\cdots n)}. \end{aligned}$$

But then we may assume here that:

$$\ell_n < \tfrac{1}{d-n-2}\left\{n(d-1)+\ell_1-\ell_n+\textstyle\sum_{i=1}^{n-1}(\ell_i-\ell_n)\right\},$$

since otherwise the right-hand side majorant can be replaced by 0, and consequently, because it follows by exponentiation from such a restriction on ℓ_n that:

$$\ell_n^{\beta_n} \leqslant \mathsf{Constant}_{n,d}\cdot \sum_{\beta'_1+\cdots+\beta'_{n-1}\leqslant\beta_n}(\ell_1-\ell_2)^{\beta'_1}\cdots(\ell_{n-1}-\ell_n)^{\beta'_{n-1}},$$

we conclude that whenever $|\ell| \geqslant 1+2n^2+(n+1)(d-n-2)$, one has:

$$\boxed{\begin{aligned} h^q\big(X,\, \mathscr{S}^{(\ell_1,\dots,\ell_n)}T_X^*\big) \leqslant \mathsf{Constant}_{n,d}\cdot \prod_{1\leqslant i<j\leqslant n}(\ell_i-\ell_j)\Bigg[\sum_{\beta'_1+\cdots+\beta'_{n-1}=n}(\ell_1-\ell_2)^{\beta'_1}\cdots(\ell_{n-1}-\ell_n)^{\beta'_{n-1}}\Bigg]+ \\ +\mathsf{Constant}_{n,d}\big(1+|\ell|^{\frac{n(n+1)}{2}-1}\big) \qquad {\scriptstyle (q=1,2\cdots n)}. \end{aligned}}$$

Application: cohomology control for $\mathscr{E}_{\kappa,m}^{GG}T_X^*$. Now, we make the following observation: no Schur bundle $\mathscr{S}^{(\ell_1,\dots,\ell_n)}T_X^*$ for which $|\ell|<\frac{m}{\kappa}$ can appear in the decomposition of $\mathsf{Gr}^\bullet\mathscr{E}_{\kappa,m}^{GG}T_X^*$ provided by the theorem on p. 30, just because all the integers λ_i^j filling the Young diagram $\mathsf{YD}_{(\ell_1,\dots,\ell_n)}$ satisfy all $1\leqslant\lambda_i^j\leqslant\kappa$, whence:

$$|\ell|\leqslant m\leqslant\kappa\,|\ell|$$

always. Thus, if we assume only that $\frac{m}{\kappa}$ is larger than the above constant $1+2n^2+(n+1)(d-n-2)$, and we certainly can assume this since both $m\gg\kappa$ and $\kappa\gg n$ are supposed to tend to infinity, then the cohomology majoration boxed above can be applied to *all* Schur bundles entering the decomposition of $\mathsf{Gr}^\bullet\mathscr{E}_{\kappa,m}^{GG}T_X^*$.

We are thus now in a position to accomplish the final series of inequalities. For any $q=1,2,\dots,n$, reminding Sections 8, 9 and 10, we have:

$$\begin{aligned}
h^q\big(X,\,\mathscr{E}_{\kappa,m}^{GG}T_X^*\big) &\leqslant \sum_{\ell_1\geqslant\ell_2\geqslant\cdots\geqslant\ell_n\geqslant 0} M_{\ell_1,\ell_2,\dots,\ell_n}^{\kappa,m}\cdot h^q\big(X,\,\mathscr{S}^{(\ell_1,\ell_2,\dots,\ell_n)}T_X^*\big)\\
&\leqslant \mathsf{Constant}_{n,d}\sum_{\substack{\mathsf{YT\;semi-standard}\\ \mathsf{weight(YT)}=m}}\ \prod_{1\leqslant i<j\leqslant n}\big(\ell_i(\mathsf{YT})-\ell_j(\mathsf{YT})\big)\bigg\{\\
&\qquad\qquad\bigg\{\sum_{\beta_1'+\cdots+\beta_{n-1}'=n}\big(\ell_1(\mathsf{YT})-\ell_2(\mathsf{YT})^{\beta_1'}\cdots\big(\ell_{n-1}(\mathsf{YT})-\ell_n(\mathsf{YT})\big)^{\beta_{n-1}'}\bigg\}+\\
&\quad+\mathsf{Constant}_{n,d}\sum_{\substack{\mathsf{YT\;semi-standard}\\ \mathsf{weight(YT)}=m}}\ \sum_{\alpha_1+\cdots+\alpha_n\leqslant\frac{n(n+1)}{2}-1}\ell_1(\mathsf{YT})^{\alpha_1}\cdots\ell_n(\mathsf{YT})^{\alpha_n}\\
&\leqslant \mathsf{Constant}_{n,d}\sum_{\mathsf{YT}\in\mathsf{YT}_{\kappa,m}^{\max}}\ \sum_{\alpha_1'+\cdots+\alpha_{n-1}'=\frac{n(n+1)}{2}}\big(\ell_1(\mathsf{YT})-\ell_2(\mathsf{YT})\big)^{\alpha_1'}\cdots\big(\ell_{n-1}(\mathsf{YT})-\ell_n(\mathsf{YT})\big)^{\alpha_{n-1}'}+\\
&\quad+\mathsf{Constant}_{n,d,\kappa}\cdot m^{(\kappa+1)n-2}\\
&\leqslant \mathsf{Constant}_{n,d}\,\frac{m^{(\kappa+1)n-1}}{((\kappa+1)n-1)!\;(\kappa!)^n}\sum_{\alpha_1'+\cdots+\alpha_{n-1}'=\frac{n(n+1)}{2}}\bigg\{\\
&\quad\bigg\{\sum_{\mu_l^i\in\nabla_{n,\kappa}}(\kappa!)^n\cdot\frac{N_{\mu_1^1}^{\kappa}}{\kappa\cdots\mu_1^1}\cdot\frac{N_{\mu_1^2,\mu_2^2}^{\mu_1^1,\kappa}}{(\kappa+\mu_1^1)\cdots(\mu_2^2+\mu_1^2)}\cdots\\
&\quad\cdots\frac{N_{\mu_1^{n-1},\dots,\mu_{n-2}^{n-1},\mu_{n-1}^{n-1}}^{\mu_1^{n-2},\dots,\mu_{n-2}^{n-2},\kappa}}{(\kappa+\mu_{n-2}^{n-2}+\cdots+\mu_1^{n-2})\cdots(\mu_{n-1}^{n-1}+\mu_{n-2}^{n-1}+\cdots+\mu_1^{n-1})}\cdot\frac{N_{\mu_1^{n},\dots,\mu_{n-1}^{n},\mu_{n-1}^{n}}^{\mu_1^{n-1},\dots,\mu_{n-1}^{n-1},\kappa}}{(\kappa+\mu_{n-2}^{n-2}+\cdots+\mu_1^{n-2})\cdots(\mu_{n-1}^{n-1}+\mu_{n-2}^{n-1}+\cdots+\mu_1^{n-1})}\cdot\\
&\quad\cdot\big[\log(\kappa)-\log(\mu_1^1)\big]^{\alpha_1'}\big[\log(\kappa+\mu_1^1)-\log(\mu_2^2+\mu_1^2)\big]^{\alpha_2'}\cdots\\
&\quad\cdots\big[\log(\kappa+\mu_{n-2}^{n-2}+\cdots+\mu_1^{n-2})-\log(\mu_{n-1}^{n-1}+\mu_{n-2}^{n-1}+\cdots+\mu_1^{n-1})\big]^{\alpha_{n-1}'}\bigg\}+\\
&\quad+\mathsf{Constant}_{n,d,\kappa}\cdot m^{(\kappa+1)n-2}
\end{aligned}$$

$$\leqslant \mathsf{Constant}_{n,d}\,\tfrac{m^{(\kappa+1)n-1}}{((\kappa+1)n-1)!\,(\kappa!)^n}\sum_{\alpha_1'+\cdots+\alpha_{n-1}'=\frac{n(n+1)}{2}}\Delta_{n,\kappa}^{\alpha_1',\dots,\alpha_{n-1}',0}+$$
$$+\,\mathsf{Constant}_{n,d,\kappa}\cdot m^{(\kappa+1)n-2}$$
$$\leqslant \mathsf{Constant}_{n,d}\,\tfrac{m^{(\kappa+1)n-1}}{((\kappa+1)n-1)!\,(\kappa!)^n}+\mathsf{Constant}_{n,d,\kappa}\cdot m^{(\kappa+1)n-2}.$$

Lastly, in the trivial minoration:

$$h^0 \geqslant \chi - h^2 - h^4 - h^6 - \cdots$$

for $\mathscr{E}_{\kappa,m}^{GG}T_X^*$, we may apply the majorations just obtained with q even and deduce that:

$$h^0\big(X,\,\mathscr{E}_{\kappa,m}^{GG}T_X^*\big)\geqslant\chi\big(X,\,\mathscr{E}_{\kappa,m}^{GG}T_X^*\big)-\mathsf{Constant}_{n,d}\,\tfrac{m^{(\kappa+1)n-1}}{((\kappa+1)n-1)!\,(\kappa!)^n}(\log\kappa)^0-$$
$$-\,\mathsf{Constant}_{n,d,\kappa}\cdot m^{(\kappa+1)n-2},$$

so that we even get a better minoration of h^0 than the one stated in the Main Theorem. □

§13. Speculations about Demailly-Semple jet differentials

Demailly-Semple invariant jet differentials. The group G_κ of κ-jets at the origin of local reparametrizations $\phi(\zeta)=\zeta+\phi''(0)\,\frac{\zeta^2}{2!}+\cdots+\phi^{(\kappa)}(0)\,\frac{\zeta^\kappa}{\kappa!}+\cdots$ of $(\mathbb{C},0)$ that are tangent to the identity, namely which satisfy $\phi'(0)=1$, may be seen to act linearly on the $n\kappa$-tuples of jet variables $\big(f_{j_1}',f_{j_2}'',\dots,f_{j_\kappa}^{(\kappa)}\big)$ by plain matrix multiplication, *i.e.* when we set $g_i^{(\lambda)}:=\big(f_i\circ\phi\big)^{(\lambda)}$, a computation applying the chain rule gives for each index i:

$$\begin{pmatrix} g_i' \\ g_i'' \\ g_i''' \\ g_i'''' \\ \vdots \\ g_i^{(\kappa)} \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \phi'' & 1 & 0 & 0 & \cdots & 0 \\ \phi''' & 3\phi'' & 1 & 0 & \cdots & 0 \\ \phi'''' & 4\phi'''+3\phi''^2 & 6\phi'' & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi^{(\kappa)} & \cdots & \cdots & \cdots & \cdots & 1 \end{pmatrix}\begin{pmatrix} f_i'\circ\phi \\ f_i''\circ\phi \\ f_i'''\circ\phi \\ f_i''''\circ\phi \\ \vdots \\ f_i^{(\kappa)}\circ\phi \end{pmatrix}\qquad{\scriptstyle (i=1\cdots n)}.$$

By definition (*see* [11, 35, 29, 17]), Demailly-Semple invariant jet polynomials $\mathsf{P}\big(j^\kappa f\big)$ satisfy, for some integer m:

$$\mathsf{P}\big(j^\kappa g\big)=\mathsf{P}\big(j^\kappa(f\circ\phi)\big)=\phi'(0)^m\cdot\mathsf{P}\big((j^\kappa f)\circ\phi\big)=\mathsf{P}\big((j^\kappa f)\circ\phi\big),$$

for any ϕ.

Then obviously when $\phi'(0)=1$, the algebra E_κ^n just coincides with the algebra of invariants for the linear group action represented by the group of matrices just written:

$$\mathsf{P}\big(j^\kappa g\big)=\mathsf{P}\big(\mathsf{M}_{\phi'',\phi''',\dots,\phi^{(\kappa)}}\cdot j^\kappa f\big)=\mathsf{P}\big(j^\kappa f\big),$$

with $\phi'',\phi''',\dots,\phi^{(\kappa)}$ interpreted as arbitrary complex constants. Such a group clearly has dimension $\kappa-1$.

This group of matrices is a subgroup of the full unipotent group, hence it is *non-reductive*, and for this reason, it not immediate to deduce finite generation,

valid in the so developed invariant theory of reductive actions, from Hilbert's averaging Reynold operator trick. Moreover, though the invariants of the full unipotent group are well understood (cf. Section 4), as soon as one looks at a *proper* subgroup of it, formal harmonies happen to be rapidly destroyed.

Three challenging questions about effectiveness that are, though, only preliminary. If one prefers to work with Demailly-Semple jets (instead of working with plain Green-Griffiths jets), then in order to reach the first stage which would correspond to knowing the exact Schur bundle decomposition for $\mathscr{E}_{\kappa,m}^{DS}T_X^*$ (instead of the one for $\mathscr{E}_{\kappa,m}^{GG}T_X^*$ provided by the theorem stated at the end of Section 4), one would have to answer *in an effective way* the following three challenging questions, for which, step by step, we explain why one should not be naive as a platonist-structuralist about what it really means to answer a mathematical question.

Question 1: *Is the Demailly-Semple algebra finitely generated?*

More precisely: is the algebra of *bi-invariant* ([29]) jet polynomials finitely generated? However, contrary to what is sometimes believed, knowing that something is "finite in cardinal" is closer to ignorance than to real knowledge, mathematically-ontologically speaking, and as an instance of this philosophical claim, it would be absolutely useless to know that the algebra of Demailly-Semple jet polynomials is generated, as an algebra, by a certain huge number, say $\leqslant 2^{2^{n\kappa}}$, of basic jet polynomials. For *effective* applications to the Green-Griffiths conjecture, one would in fact need to know not only the *exact minimal* number for such a system of generators, but also the *weight* of each a generator, and even *all the generators themselves*. However and most importantly, this would even not at all be enough, as the second obstacle comes immediately.

Question 2: *Is the ideal of relations between a set of basic generating Demailly-Semple invariants finitely generated?*

Again, in order to be able to describe the *exact* Schur bundle decomposition as was done in Section 4, it is absolutely necessary to describe effectively and for arbitrary n, κ the ideal of relations. For jets of order $\kappa = 4$ in dimension $n = 4$, we were unable to describe the full ideal of relations between the 2835 basic generating invariants listed in [29], not to mention that we ignore what is the minimal number of generators. We were saved in [29] by the fact that there are "only" 16 basic bi-invariants (minimal number) and "only" 41 relations between them[28] (in a Gröbner basis for a certain lexicographic order).

All these speculative considerations lead us *in fine* to the main metaphysical question: *Are there observable, simple mathematical harmonies in a certain set of generators and for all the relations between them?* Without harmonies, there is absolutely no hope to treat the case where n and κ are arbitrary. For $n = 2$, $\kappa = 5$ and for $n = 4$, $\kappa = 4$, we were unable, in [29], to discover any

[28] We believe that one could attack the seemingly accessible case $n = \mathbf{5}$, $\kappa = \mathbf{5}$.

combinatorially convincing global formal harmonies. Nevertheless, there could yet be some slight hope as follows.

Question 3: *Is the algebra of bi-invariants Cohen-Macaulay?*

Well, this would be nice. For reductive group actions, this is known to be true, but however, *almost never in a neat effective way*. At least, one could dream that the Demailly-Semple algebra is Cohen-Macaulay and that a basis of so-called *primary invariants* presents some understandable harmonies. It is known, then, that the effective calculations about Euler-Poincaré characteristic and cohomologies with an adapted reduced Schur bundle decomposition become much more tractable when one looks only at primary invariants. But for $n = 4$, $\kappa = 4$ and for $n = 2$, $\kappa = 5$, going through the mutually independent bi-invariant we exhibited in [29] and trying to change the generators, we were unable to see or to devise a neat basis of primary invariants, though for $n = 2$, $\kappa = 4$, one easily discovers such a basis at first glance. Again, a non-effective theorem claiming "the algebra of Demailly-Semple is Cohen-Macaulay" would be useless toward the Green-Griffiths conjecture because rather, one would really need to know the exact description of a basis of primary invariants with all their weights in order to start continuing working toward the Green-Griffiths conjecture. But if the algebra is not even Cohen-Macaulay, well, the next tasks could be even more extremely challenging because, as we already saw, the end of Section 4 opens several doors to other fields of hard effective computations when one just deals with the much simpler Green-Griffiths jets.

Last but not least, we would like to insist on the fact that in the state of affairs which is current since the 19th Century, even for the most studied reductive action of $\mathsf{SL}_2(\mathbb{C})$ on binary forms of degree d in (only) two variables, the *effective answers* to Questions 1, 2 and 3 is unknown for arbitrary d, and is rather extremely challenging in fact. Cayley, Sylvester, Gordan, Noether, Popov, Grosshans, Springer, Dixmier, Lazard, Bedratyuk and others did not find any complete closed global tamed combinatorial harmonies valid for arbitrary d.

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